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ENGINEERING CONSTRUCTION

IN

STEEL AND TIMBER

BY

WILLIAM HENRY WARREN

WHITWORTH SCHOLAR; MEMBER OF THE INSTITUTION OF CIVIL ENGINEERS; MEMBER OF THE
AMERICAN SOCIETY OF CIVIL ENGINEERS; MEMBER OF COUNCIL OF THE INTERNATIONAL
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PREFACE TO THE SECOND EDITION.

THE development in the science of testing materials and the various researches undertaken during the last fifteen years have supplied an enormous amount of data on the physical properties of the materials of construction. Wrought iron has been replaced by structural steel. The safe intensity of working stresses in structures subjected to moving train loads is now considered to be best determined by means of an equivalent dead load, equal to the live load increased by an amount representing the impact effect, added to the dead load, in preference to the so-called fatigue formulæ for working stresses. It has therefore been considered necessary to re-write the first two chapters.

Additional matters have been dealt with in Chapter IV. on influence lines, in connection with the bending moments and shears.

Again, in Chapter VI. more modern examples of timber structures have been worked out.

In Chapter X. the deflections due to shearing stresses have been considered, also the deflection of trusses and girders.

In Chapter XI. influence lines have been used in connection with continuous girders, and the design of cantilever bridges has been more fully considered.

The strength of columns in Chapter XII. has been treated in a more complete manner, consistent with the importance of the subject, and the portion of Chapter XIII. on riveted joints has been re-written.

Chapter XIV. has also been re-written.

Chapter XVI. is entirely new, and deals with the modern methods of considering the loads on bridges.

The examples worked out in Chapters XVII. to XIX. have been most carefully considered, and may be regarded as repre-

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senting the most modern practice in the design of plate-web girders, and trusses for railway and highway bridges.

Chapter XX. on swing bridges has been extended in order to include some well-known types of this class of bridge built in America.

Chapter XXI. on arched and suspension bridges has also been considerably extended, and the analytical treatment of the subject has been considered with the aid of influence lines and various tables to facilitate the calculations in these types of structures.

It is hoped that the book, in its present form, will be found more generally useful to the student and the engineer.

The author desires to acknowledge his obligations to the American Bridge Company for the designs of the plate web girder, the Pratt, and Parker truss bridges, and to Mr. W. J. Hanna, Under Secretary of Public Works, New South Wales, for the design of the Pratt truss highway bridge, also to Mr. G. E. Cowdery, B.E., for his assistance in connection with the reading and correction of the proofs.

W. H. W.

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CHAPTER I.

STRENGTH, ELASTICITY, ENDURANCE, AND SAFE WORKING STRESS IN IRON AND STEEL.

THE strength of a structure, such as a roof or a bridge, depends not only upon its form and dimensions, but also upon the material used in its manufacture. The load or loads upon a structure produce stresses in the various members, which may be tensile, compressive, shearing, and occasionally torsional; the stresses develop resistances in the materials which are generally accompanied by slight alterations in form, termed deformations, such as elongation, shortening, or deflection of the member in question. The deformation, whether elongation, shortening, or deflection, is termed the strain, which must not be confounded with the stress producing it. The stresses produced in the various parts of a structure depend upon the form and dimensions of the structure and the loads which it carries; but in arranging the cross-section and dimensions of the various members to resist the stresses developed in them, it is necessary to know the physical properties of the materials used, such as the tensile, compressive, and shearing strengths, the elasticity, ductility, rate of expansion by heat, etc. In order to design a structure in an economical manner we require to know many other things, such as, in the case of steel, the ordinary and maximum sizes of plates, bars, angles, channels, and other sections, which are produced commercially, and their relative cost; how best to connect the various members together so that they may have the necessary strength, and otherwise fulfil their purpose efficiently; what portions are best made of cast-iron, and how such castings should be designed. If stone, brick, concrete, reinforced or plain, is used in the structure, such as for abutments and piers of bridges, it will be necessary to know how to dispose these materials to resist the stresses developed in them, and how best to limit the pressure on foundations, depending upon the nature of the ground, whether rock, clay, sand, or other material upon which the structure is to be built. It follows, therefore, that a sound knowledge of materials is the basis of the economical design of structures. In the first place, it is proposed to deal with those physical properties of the chief materials used in structures which it is essential that an engineer should know, whose business it is to design such structures, more especially with structural steel, cast-iron, and timber. Our knowledge of materials is derived from investigations conducted in laboratories by means of testing machines and appliances, and these exist in great variety in different countries. It is essential in all such appliances to be able to apply a definite stress, and determine the deformation or strain produced by the stress with sufficient accuracy, consistent with the object of the particular test in question. It is not proposed to describe such appliances, or to discuss the methods of testing materials, as these are best studied in works devoted to these matters,[1] but the results of testing will be considered in so far as they affect the design of structures.

Tensile Stress and Strain.—If two equal forces P act on a long prismatic body in the direction of its longitudinal axis, we may assume that the forces are uniformly distributed over the cross-section of the body of area, a, then the stress, or intensity of stress, σ, is—

$$\sigma = \frac{P}{a}.$$

The stress may be expressed in pounds or tons per square inch, or in kilograms per square millimeter or centimeter. Under the effect of tensile stress the body elongates, and at the same time the cross-sectional area diminishes. Let l denote the original length, and l_1 the length after stretching, then—

$$l_1 - l = \text{the elongation or strain.}$$

If λ denote the elongation per unit of length, l, then—

$$\lambda = \frac{l_1 - l}{l}.$$

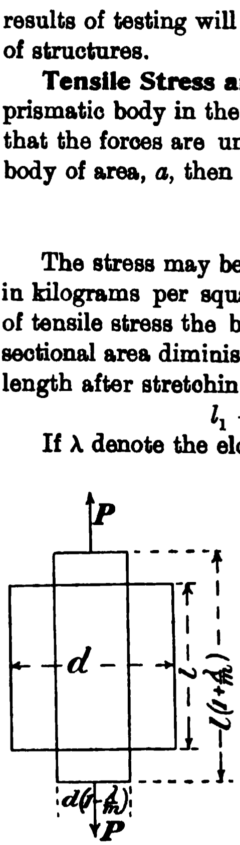

FIG. 1.

The ratio of the unit stress to the unit strain is termed "Young's Modulus," or coefficient of elasticity, generally denoted by E :—

$$\frac{\sigma}{\lambda} = \text{E}.$$

The body must contract laterally in consequence of its extension longitudinally; if the density remains constant, and it is found that there is a definite ratio known as "Poisson's Ratio" denoting the relationship between the extension and corresponding contraction $\frac{1}{m}$, the unit contraction is $\frac{\lambda}{m}$. In metals m varies from 3 to 4; for india-rubber, m is about 2, provided that the ratio $\frac{\sigma}{\lambda}$ remains constant. Fig. 1.

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[1] *Handbook of Testing Materials*, by Prof. Martens; *The Testing of the Materials of Construction*, by Prof. Unwin; *Reports of the International Society for Testing Materials.*

Limit of Proportionality between σ and λ. Yield-point.—When a long prismatic bar is subjected to tensile stress gradually increased, and the increments of elongation are accurately determined for equal increments of stress, then, if the material is structural steel, it will be found that the ratio of the stress to the strain is very approximately constant up to a certain limit, termed the "limit of proportionality," or the elastic limit, after which the strains increase more rapidly than the stresses producing them, until a point is reached, generally very near to the limit of proportionality, where the strain increases without increase of stress, termed the "yield-point". The yield-point is indicated in testing machines of the lever type by the drop of the lever. The yield-point is frequently referred to as the elastic limit of the material, and there is much confusion in the use of these terms in reports of tests of material. The true elastic limit is the stress which will just not produce a permanent deformation found by repeated loading and unloading, and it is not coincident with the limit of proportionality. If equal increments of load are applied with an interval of say 5 minutes between each, and the extensions are determined in $\frac{1}{10000}$ of a millimeter as with Martens' Mirror Extensometer,[1] it will be found that the results obtained plotted with σ as abscissæ, and λ as corresponding ordinates, do not lie exactly in a straight line, but on a flat curve. A tangent drawn at the origin, and cut by horizontal lines parallel to the abscissæ, show that part of this so-called elastic deformation is really plastic. In structural steel, and in annealed tool steel, the elastic limit, limit of proportionality, and the yield-point can be determined by proper methods, but in hardened tool steel, and in some varieties of cast-iron, there is no yield-point; also in the latter every load produces a permanent set and there is no defined limit of proportionality. In brickwork, cement, mortar, or concrete every load produces a permanent deformation and the coefficient of elasticity is not a constant, but if we denote the small increments of stress by $\delta\sigma$, and the corresponding increment of strain by $\delta\lambda$, we may write for the particular stress in question—

$$\frac{\delta\sigma}{\delta\lambda} = \mathrm{E}.$$

The yield-point has been described as the commencement of the large plastic deformations, and these increase in a still greater ratio as the stress is increased up to that producing fracture. The actual load-producing fracture in ductile materials, such as structural steel, is always less than the maximum load sustained just before fracture, as local stretching or necking occurs about the region where fracture actually occurs. It is usual to obtain an autographic record of the results of testing, called a stress-strain diagram, which should represent the characteristic properties of the material, but the scale is usually too small to show clearly the behaviour of the material within the limit of proportionality. Fig. 2 shows such a diagram for structural steel, where p denotes the limit of proportionality; y, the yield-point; mg, the maximum load sustained; bt, the breaking load; og, the general, and gt the local extension; also the total extension is shown by ot.

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[1] *Handbook of Testing Materials*, by Prof. Martens.

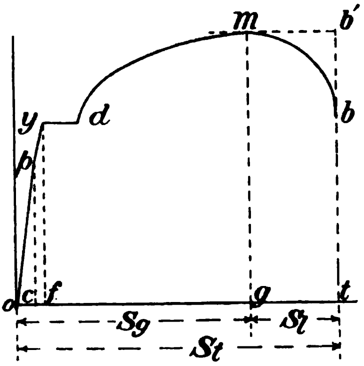

FIG. 2.

Table I. gives the results obtained from testing pieces of structural steel such as would be used in roofs and bridges. The test-sheet consists of 21 columns, of which the first five need no explanation Column 6 gives the total load necessary to produce fracture corresponding with mg, Fig. 2. Column 7 is found by dividing the result in column 6 by the original cross-sectional area of the test-piece in square inches. Column 8 is found by dividing column 7 by 2240. Column 9 is found from Fig. 2 and represents cp expressed in pounds. Column 10 is the ratio of the results in columns 9 and 6 × 100. Columns 11, 12, and 13 are obtained from measurements of the dimensions of the fractured area after testing. Column 14 is found thus:—

Let a denote the original, and a' the fractured area, c the contraction of area, then—

$$c = a - a',$$

or expressed in per cent. of original area—

$$c\,\%_{\circ} = 100\left(1 - \frac{a'}{a}\right).$$

Columns 15, 16, 17, 18, and 19 refer to elongations measured on lengths of the test-piece, usually 8 in. and 4 in. in standard bars; the figures in the remaining columns 17 to 19 are found as follows:—

Let λ_g denote the elongation per in. on the gauge length, up to the maximum load sustained after which necking or local elongation occurs; also let l_8 and l_4 denote the elongations measured after fracture on the gauge lengths of 8 and 4 in. respectively; let λ_l denote the local extension from m to b, Fig. 2; then—

$$l_8 = \lambda_l + 8\lambda_g \qquad (1)$$
$$l_4 = \lambda_l + 4\lambda_g \qquad (2)$$

TABLE I.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Test Number.	Description.	Original Dimensions.			Stress in Pounds		Stress in Tons, per sq. Inch.	Apparent Limit of Elasticity from Auto-diagrams. Pounds.	Ratio of Limit to Break, per Cent.	Contracted Dimensions.			Contraction of Area, per Cent.	Elongations measured after fracture.		Local Elongation, Inches.	General Elongation, per Cent.	Total Elongation, per Cent. on 8 in.	Coefficient of Quality.	Remarks.
		Breadth, Inches.	Thickness, Inches.	Area.	Total.	Per sq. Inch.				Breadth, Inches.	Thickness, Inches.	Area.		On 8 in.	On 4 in.					
1	Flat bar cut from plate	2·0	0·372	0·744	49,000	65,860	29·4	28,000	57·0	1·52	0·26	0·395	47·0	2·15	1·40	0·65	18·7	26·8	—	Sound clean fracture.
2	Do. do.	2·0	0·340	0·680	45,500	66,900	29·8	29,000	63·8	1·53	0·24	0·367	46·0	2·35	1·46	0·57	22·2	29·3	—	Do. do.
3	Do. do.	2·21	0·250	0·553	35,500	64,253	28·6	26,250	73·9	1·65	0·769	0·279	49·5	2·04	1·37	0·70	16·7	25·5	—	Do. do.
4	Do. do.	2·21	0·253	0·559	37,000	66,174	29·5	28,500	77·0	1·71	0·780	0·308	44·9	2·35	1·70	1·05	16·2	29·4	—	Do. do.
5	Do. do.	2·12	0·241	0·511	32,250	63,121	28·2	24,400	75·6	1·57	0·15	0·235	53·9	1·99	1·28	0·57	17·7	24·9	—	Do. do.
6	Do. do.	1·80	0·530	0·954	60,400	63,312	28·3	41,000	67·9	1·38	0·39	0·538	48·6	1·96	1·34	0·72	15·5	24·5	—	Do. do.
7	Do. do.	2·12	0·245	0·519	32,250	62,091	27·7	24,300	75·4	1·63	0·12	0·196	62·3	1·80	1·10	0·40	17·5	22·5	—	Do. do.
8	Do. do.	1·51	0·489	0·738	52,700	71,408	31·8	32,500	61·7	1·15	0·35	0·403	45·5	2·03	1·28	0·53	18·7	25·4	—	Do. do.
9	Round bar.	Dia.	0·603	0·286	20,500	71,784	32·1	12,000	58·5	Dia.	0·43	0·145	49·2	2·09	1·33	0·57	19·0	26·2	—	Do. do.
10	Flat plate cut from angle	2·05	0·242	0·496	30,250	61,000	27·2	24,800	76·3	1·60	0·16	0·256	48·4	2·30	1·40	0·50	22·5	28·8	—	Do. do.

From these two equations we can find λ_l and λ_g. Hence the general extension per unit of length is—

$$\lambda_g = \frac{l_8 - l_4}{4}, \text{ and } \lambda_l = l_8 - 8\lambda_g.$$

λ_l is inserted in column 17, and $100\frac{\lambda_g}{l_8}$ is inserted in column 18. Column 19 gives the values of the total extension on the gauge length 8 in. expressed in percentage of the original length, *i.e.*, $\frac{100 l_8}{8}$. In order to determine the elongations produced on any portion of the gauge length, such as 4 in., it is necessary to accurately divide the test-piece before testing as shown in Figs. 3 and 4, which represent standard test-bars as used in the author's laboratory.

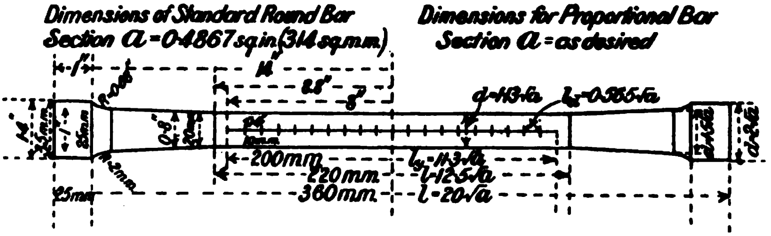

FIG. 3.

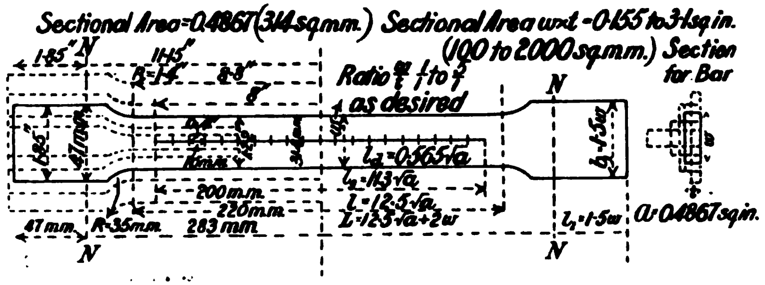

FIG. 4.

The assumption made in calculating the general and local extensions by the foregoing method, is that the extension is constant over the length included between the gauge marks up to the point when the maximum load is sustained; this is not strictly true for test-pieces of the lengths and proportions ordinarily used owing to the effect of the grips. It is however worth while to separate, even approximately, the general from the local extension. If σ_m denotes the maximum stress sustained, and λ_g the total elongation per unit of length, the product $\sigma_m \lambda_g$ has been proposed by the late Prof. Tetmajer as a coefficient of quality; thus, if $\sigma_m = 30$ tons per sq. in., and $\lambda_g = 0{\cdot}22$ in., then the quality factor would be 6·6.

The elongation per cent. on the gauge length and the contraction of area per cent. are used to express the ductility of the material; the former is more usual in modern specifications. The elongation is related to the contraction of area at fracture in a perfectly plastic material if the volume remains constant; thus—

$$al = a'(1 + \lambda)l \ \therefore \ \frac{a'}{a} = \frac{1}{1 + \lambda}.$$

The contraction of area is—

$$a - a' = a\left(1 - \frac{1}{1 + \lambda}\right) = a\frac{\lambda}{1 + \lambda},$$

or proportional to the elongation.

During plastic extension the stress is constant, so that the load diminishes as the bar elongates.

Elastic Resilience.—When a piece of material is stressed up to the limit of proportionality as illustrated in the triangle *opc* (Fig. 2), the area of the triangle represents the work done, or the energy stored in the test-piece, capable of being restored on the release of the stress. Since $\lambda = \frac{\sigma}{E}$, the area of the triangle represents the elastic resilience of the piece per unit of volume:—

$$\frac{\sigma^2}{2E}.$$

The work of resilience to be done up to the yield-point, although this energy cannot be restored, is represented by the area of the figure *opyfo*, and up to the maximum load applied by the area *opydmg*. The part of the resilience overcome beyond the maximum stress is of no consequence to the engineer. The area of the diagram *opydmbt* ÷ *al* is a measure of the resilience which a unit volume of the material opposes to rupture, and is sometimes used to express the quality of the material.

Stress and Strain in Compression.—The relations and ideas in crushing tests are similar to those of the tension test, only the forces must be considered as acting in the reverse direction. The compressive stress on a prism of length l and cross-sectional area a subjected to a force P is—

$$-\frac{P}{a} = -\sigma.$$

The shortening or upsetting per unit of length is—

$$-\frac{l - l_1}{l} = -\lambda.$$

The coefficient of elasticity in crushing is—

$$E = \frac{-\sigma}{-\lambda} = \frac{\sigma}{\lambda}.$$

The shortening per cent. is—

$$-100\lambda = -100\left(\frac{l - l_1}{l}\right).$$

There is a proportional elastic limit for crushing similar to that for tension in many materials such as steel. Poisson's ratio $\frac{1}{m}$ has the same values as in tension tests but with the sign reversed. The elastic resilience is $\sigma^2/2E$ as in tension tests, and the total resilience is the work done up to the yield-point, or to the crushing stress.

The yield-point is more or less well defined in crushing tests, but failure by crushing is only clearly defined in brittle materials, such as cast-iron, bricks, concrete, stone, etc., whereas in ductile materials such as soft lead, steel, copper, etc., fracture cannot occur, as they fail by plastic yielding without showing any signs of fracture. It is, therefore, customary to use the yield-point in the crushing tests of ductile materials, and the actual crushing stress in brittle materials.

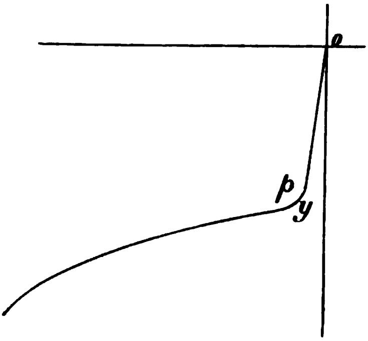

FIG. 5.

Fig. 5 shows a stress-strain diagram for a crushing test, and the effect produced by the stress is to cause a bulging of the test-piece. In making crushing tests with short cylinders they tend to assume a barrel form. If l_g denotes the original length of the test-piece and d the diameter, also l and d' the length and diameter respectively produced by the load, then—

The shortening is $l_g - l$; the area a found from the original area a_o is—

$$a = a_o \frac{l_g}{l}.$$

The diameter d' is found from a:—

$$d' = \sqrt{\frac{4a}{\pi}}.$$

The increase in diameter per unit of diameter:—

$$\lambda = \frac{d - d'}{d}.$$

The stress on the compressed area if Q denotes the total load applied is—

$$\sigma = \frac{Ql}{a_o l_g}.$$

The maximum stress in the material, or the pressure of plasticity, or fluidity, occurs when σ remains constant after the total load has been increased; the bulging of the cylinder will be sufficient to cause the intensity of σ to be constant.

Shearing Stress.—Fig. 6 shows a prism subjected to a compressive force P acting along the axis. The intensity of stress on a cross-section AB of area equal to a is—

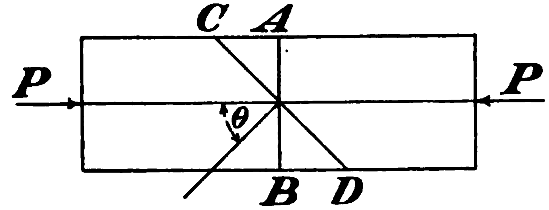

FIG. 6.

$$\sigma = \frac{P}{a}.$$

The total stress on an oblique section CD whose normal is inclined to the axis at an angle denoted by θ may be resolved normally and tangentially, thus:—

$$P_n = P\cos\theta;\ P_t = P\sin\theta.$$

Dividing these by the area of the section CD $= a' = a/\cos\theta$ we have the intensities of normal and tangential stress:—

$$\sigma_n = \frac{P\cos\theta}{a'} = \frac{P}{a}\cos^2\theta = \sigma\cos^2\theta.$$

$$\sigma_t = \frac{P\sin\theta}{a'} = \frac{P}{a}\sin\theta\cos\theta = \frac{\sigma\sin 2\theta}{2}.$$

σ_t is a maximum when $\sin 2\theta = 1$, and $\theta = \frac{\pi}{4}$.

σ_t is called the shearing stress, and its maximum value is—

$$\sigma_t = \sigma\sin\frac{\pi}{4}\cos\frac{\pi}{4} = \frac{\sigma}{2}.$$

When a prism is subjected to a simple push or pull in a testing machine, it frequently happens that fracture occurs by shearing along a plane inclined to the direction of the applied forces.

Combination of two Simple Stresses acting at Right Angles to each other.—Fig. 7 represents a prism *abcd* subjected to a simple pull P_y acting along the axis, and also to a simple pull P_x acting at right angles to P_y in the centre of the opposite faces *ad* and *bc*, giving rise to stresses σ_y and σ_x respectively, uniformly distributed over the two pairs of faces.

Let an oblique plane EF be inclined to P_x at an angle θ, and to P_y at an angle $\frac{\pi}{2} + \theta$.

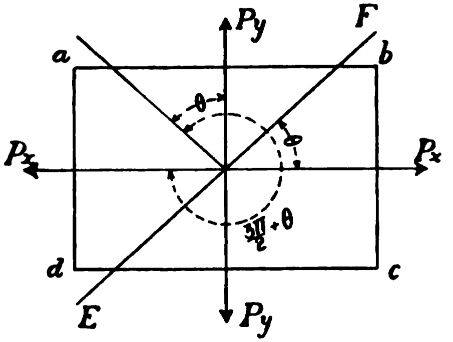

Fig. 7.

Then the intensity of normal stress σ_n on EF due to P_y and P_x is—

$$\sigma_n = \sigma_y \cos^2\theta + \sigma_x \cos^2\left(\frac{3\pi}{2} + \theta\right) = \sigma_y \cos^2\theta + \sigma_x \sin^2\theta.$$

The intensity of tangential stress σ_t on EF is—

$$\sigma_t = \sigma_y \sin\theta\cos\theta + \sigma_x \sin\left(\frac{3\pi}{2} + \theta\right)\cos\left(\frac{3\pi}{2} + \theta\right)$$

$$= (\sigma_y - \sigma_x)\sin\theta\cos\theta.$$

σ_t is a maximum when $\theta = \frac{\pi}{4}$ $\therefore$ σ_t (max.) $= \frac{\sigma_y - \sigma_x}{2}$, also σ_n (max.) $= \frac{\sigma^y + \sigma_x}{2}$.

If $\sigma_y = -\sigma_x$, that is to say if the stresses are equal in magnitude but opposite in sign, and $\theta = \frac{\pi}{4}$, then—

$$\sigma_n = 0, \text{ and } \sigma_t = \sigma_y = \sigma_x.$$

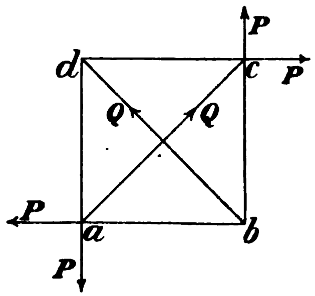

Fig. 8.

There is therefore nothing but tangential stress, or shearing stress, on the two planes at right angles to each other, and these are of equal intensity. The equality of the intensities of shearing stresses, on two planes at right angles to each other, may be shown also in the following manner. Let Fig. 8 denote a cubical block in equilibrium under forces applied at opposite faces in the manner indicated. The couple acting along *ad* and *cb* must be balanced by the couple acting along *cd* and *ab*, since they tend to cause rotation in opposite directions. If we consider the forces acting at *a*, the resultant Q is—

$$Q = \sqrt{2}P.$$

But Q is distributed over an area *db* which is $\sqrt{2}$ times greater than the face of the cube, hence the intensity of stress on the face is equal to that on the plane containing the diagonal. In a similar manner it may be shown that the forces acting at *b* develop stresses on the face and diagonal plane containing *ac*, which are of equal intensity.

Coefficient of Rigidity.—The effect of the shearing stresses in producing shearing strain is seen in a cubical element *abcd* (Fig. 9) subjected to equal shearing stresses on two pairs of opposite faces. The distortion consists of a shortening along one diagonal, and a lengthening along the other, the sides remaining parallel and the volume constant. The square *abcd* becomes *ab'c'd'*, and the angles are increased and diminished by a small amount θ to the values $\frac{\pi}{2} + \theta, \frac{\pi}{2} - \theta$. Within the limit of proportionality of the stresses and strains,

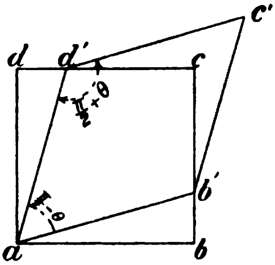

FIG. 9.

$$\theta = \frac{\sigma}{C},$$

where C is a constant for the particular material expressing its resistance to shearing or "Coefficient of Rigidity,"

$$C = \frac{\sigma}{\theta},$$

or the ratio of shearing stress to shearing strain. The value of C is best determined by experiments on torsion, and is usually about $\frac{2}{5}$E.

Shearing Tests.—Shearing tests are usually made so that two surfaces are subjected to shearing stress, such as the sections at 1 and 2 (Fig. 10).

The force P producing shearing at the two planes is assumed to be equally divided between the two sections 1 and 2, and the shearing stress is—

$$\sigma_s = \frac{P}{2a},$$

where *a* denotes the area of the section at 1 and 2. If *d* denotes the diameter of a circular pin subjected to shear in the manner indicated in Fig. 10,

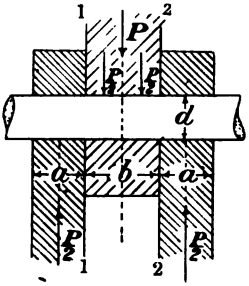

FIG. 10.

$$\sigma_s = \frac{P}{\frac{2\pi d^2}{4}} = \frac{2P}{\pi d^2}.$$

The pin is said to be in double shear since the two surfaces at 1 and 2 must shear at the same time. It will be more easily realised after the study of transverse stresses in Chaps. IV. and V. that there must always be a bending couple tending to cause deflection of the pin, and also that the distribution of the stresses over the distances *a*, *a*, and *b* cannot be uniform. It will also be seen that the thicknesses of the portions *a*, *b*, and *a* must affect the results obtained in double shear.

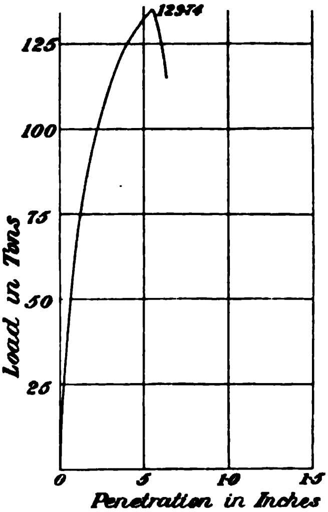

Fig. 11.

Fig. 11 shows a diagram obtained by Mr. Wicksteed's single shear apparatus from a piece of basic-steel having a section 3 in. by $2\frac{1}{2}$ in., and an area of 7·5 sq. in. The maximum load applied was 129·74 tons, equivalent to 17·3 tons per sq. in. on the original section. The amount of penetration at the maximum load was 0·55 in., and the total penetration before fracture was 0·65 in. The work done in shearing the bar was 66·25 in. tons, and the approximate shearing stress on the ultimate area was 30·9 tons per sq. in.

The diagram clearly shows that the process of shearing is gradual, and consists of two parts, compression and indentation, until the area is sufficiently reduced to make the specimen more willing to shear than to stand any further compression.

A pure shearing stress cannot be produced in the manner indicated in Fig. 10. It is possible to get over the difficulty of equal division between the two surfaces in double shear by arranging that only one surface is subjected to shearing; and many devices have been proposed and used for making single shear tests, such as the one illustrated in Fig. 12, devised by Unwin,[1] which is very satisfactory in timber tests. But in all cases there is a bending effect, which cannot be entirely eliminated.

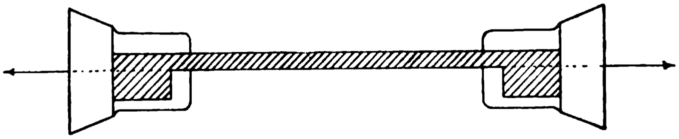

Fig. 12.

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[1] The author has rounded the square corners to concentrate the force more along the centre line and reduce the bending moment in his tests.

Punching Tests.—Punching is a similar process to shearing, only that a cylindrical surface is cut instead of a plane. The punching strength may be expressed thus (Fig. 13):—

$$\sigma_p = \frac{\mathrm{P}}{a} = \frac{\mathrm{P}}{\pi d t},$$

where a denotes the area, and t the thickness of the plate punched. In thin plates where t is small relatively to d, the punching strength is equal to the shearing strength; for thick plates the process is complex. The distribution of stress in punching may be better realised in subjecting a cylindrical block of lead to punching along its axis (Fig. 14). Then so long as the force P is not sufficient to produce shearing of the cylindrical section, the material will flow under the punch laterally and upwards. Penetration occurs only after the crushing resistance of the material has been overcome, when free flow of the material ceases.

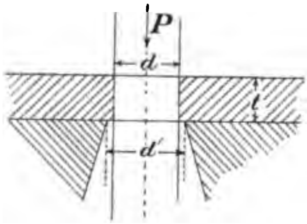

FIG. 13.

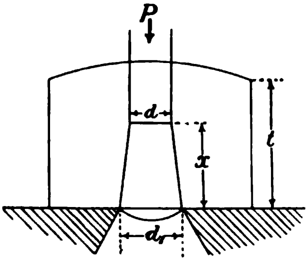

FIG. 14.

If σ_c is the stress at the crushing limit of the material, appreciable flow can only take place when

$$\mathrm{P} > d^2 \frac{\pi}{4} \sigma_c.$$

Shearing of the cylinder can only occur when the height x has been decreased to such an extent that

$$\sigma_s d \pi x < \mathrm{P},$$

where σ_s is the intensity of shearing stress. Hence the shape and thickness of the test-piece must influence the result, and punching is therefore a complex process, in which the metal under the punch becomes plastic, and flows until the remaining metal under the punch is too thin to supply the necessary shearing resistance called into play by the pressure of the punch. Tresca has shown in his experiments on plastic materials, such as lead, that the large plastic deformations did not alter the density of the material, and further that the deformations required time for their development, increasing as the time during which the stresses producing them is allowed to act.

The wad punched out from a cylindrical block of lead is reduced in volume, in consequence of the lateral flow of the metal into the block during the process of punching. The process of squirting lead pipes under hydraulic pressure is an example of the plastic flow of metals under a pressure sufficiently great to cause an indefinitely great deformation.

Torsion.—When a bar of circular cross-section is fixed at one end, and a couple is applied at the other end, the axis of the bar being the axis of the couple, and the plane of the couple at right angles to the axis of the bar, it suffers a deformation termed torsion. The stress is everywhere one of pure shear, and the strain is due to the rotation of each plane of cross-section relatively to adjacent planes. The strain along the axis is zero, and it increases in proportion to its distance from the axis.

In Fig. 15 a straight line *ab* parallel to the axis becomes a helix *ac* making an angle ϕ with *ab* on the surface of the cylinder. The radius *ob′* turns round to *oc′* through an angle θ called the angle of twist, and

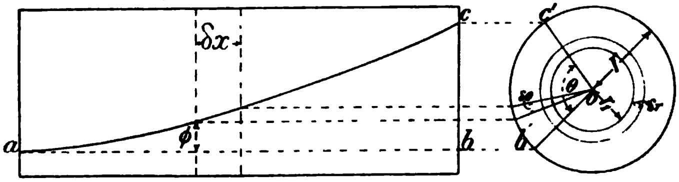

FIG. 15.

is proportional to the length *ab*. For two parallel cross-sections of the cylinder at right angles to the axis, at a distance apart δx, the angle of twist is $\delta\theta$, and

$$\phi\delta x = r\delta\theta.$$

If σ denote the intensity of shearing stress on the surface of the cylinder,

$$\sigma = c\phi = cr\frac{\delta\theta}{\delta x},$$

where *c* is the coefficient of rigidity.

The moment of resistance of the cylinder to twisting must equal the moment of the couple producing twisting. Since the intensity of shearing stress is proportional to the shearing strain within the elastic limit of the material, and is zero at the centre *o*, increasing towards the surface proportionally to the radius, let σ denote the stress at the surface, *r* the radius, and σ_1 the stress at a radius r_1; then—

$$\sigma = \frac{\sigma_1 r}{r_1}.$$

The moment of resistance of a ring having the inner and outer radii r_1 and $r_1 + \delta r_1$ respectively is equal to the area of the ring multiplied by the stress over it, and also by the radius r_1:—

$$2\sigma_1 \pi r_1{}^2 \delta r.$$

The moment of resistance of the whole section is the sum of the moments of resistance of all such rings; and if T denotes the twisting moment for the whole cross-section, we have—

$$\mathrm{T} = \int 2\pi r_1{}^2 \sigma_1 \delta r\,;$$

substituting for σ_1 and integrating between the limits r and o—

$$\mathrm{T} = \frac{2\pi\sigma}{r}\int_o^r r_1{}^3 dr = \frac{\pi}{2} r^3 \sigma.$$

For a hollow shaft, the internal and external radii being r_2 and r respectively—

$$\mathrm{T} = \frac{2\pi\sigma}{r}\int_{r_2}^r r_1{}^3 dr = \frac{\pi\sigma(r^4 - r_2{}^4)}{2r}.$$

Since the polar moment of inertia, I_p, about the axis of the bar is

$$\mathrm{I}_p = \frac{\pi r^4}{2},$$

$$\mathrm{T} = \frac{\sigma \mathrm{I}_p}{r} = \frac{\pi r^3 \sigma}{2} = \frac{\pi d^3 \sigma}{16} = 0{\cdot}196 d^3 \sigma.$$

Angle of Torsion in Circular Shafts.—

Since $\mathrm{T} = \dfrac{\sigma \mathrm{I}_p}{r}$ $\quad\therefore \sigma = \dfrac{\mathrm{T}r}{\mathrm{I}_p}$.

Let i denote the angle of torsion per unit of length, then—

$$i = \frac{\delta\theta}{\delta x} = \frac{\phi}{r} = \frac{\sigma}{cr} = \frac{2\mathrm{T}}{\pi c r^4} \qquad \therefore c = \frac{2\mathrm{T}}{\pi i r^4} = \frac{32\mathrm{T}}{\pi i d^4}.$$

In a hollow circular shaft—

$$i = \frac{2\mathrm{T}}{\pi c(r^4 - r_2{}^4)} = \frac{32\mathrm{T}}{\pi c(d^4 - d_2{}^4)}.$$

The following values of σ and c have been obtained in torsion tests:—

TABLE II.

Material.	Value of σ. $\sigma = \dfrac{\mathrm{T}}{0{\cdot}196 d^3}$ pounds, per sq. in.	Coefficient of rigidity c. $c = \dfrac{32\mathrm{T}}{\pi i d^4}$ pounds, per sq. in.
Cast-iron . . .	35,000	7,000,000
Wrought-iron . . .	55,000	12,000,000
Rivet steel . . .	67,000	13,000,000
Bessemer steel . .	100,000	12,800,000
3 per cent. nickel steel .	93,300	11,600,000
6 per cent. nickel steel .	113,700	11,200,000

Resilience of Torsion.—The resilience of torsion is the work expended in producing elastic strains in torsion. If l denote the length of the bar, li is the total angle of torsion. The differential work performed by the twisting moment T, in producing an indefinitely small twist $d(li)$, is—

$$\mathrm{T}ldi.$$

The whole work done is—

$$\text{Resilience} = \int \mathrm{T}ldi.$$

Substitute for T the value already found; then since

$$\mathrm{T} = ci\mathrm{I}_p$$

we have—

$$\text{Resilience} = c\mathrm{I}_p l\int_0^i idi = \tfrac{1}{2}c\mathrm{I}_p li^2.$$

If P denotes the force applied with leverage R, then—

$$i = \frac{\mathrm{PR}}{c\mathrm{I}_p}$$

and

$$\text{Resilience} = \frac{\mathrm{P}^2\mathrm{R}^2 l}{\pi cr^4}.$$

Transverse Tests.—The transverse tests of material usually consist in supporting a bar at each end and loading it in the centre. The bar may be prismatic or cylindrical, but is usually the former. If b denote the breadth, d the depth of the bar, and l its length between the centres of the supports, then if W denote the central load-producing fracture, it will be shown in Chap. IV. that

$$\mathrm{W} = \frac{2bd^2f}{3l}.$$

Where f is a quantity for the material under test termed the "modulus of rupture,"

$$f = \frac{3\mathrm{W}l}{2bd^2}.$$

To obtain uniform results the bars must be geometrically similar in every respect. To determine the quality of cast-iron for columns, it is usually specified that bars 1 in. wide by 2 in. deep, supported on knife edges 36 in. centres, shall sustain without fracture a central load of 3000 pounds, so that f is equal to 40,500 pounds per sq. in.

Timber, steel, and other materials are frequently tested in this manner.

If the deflection is noted for each increment of load, it will be seen that the deformations are proportional to the load up to a certain limit, and that afterwards they increase more rapidly than the loads producing them until failure is produced. In the case of brittle materials, such as cast-iron, fracture will take place, but in the case of ductile materials, such as soft steel, failure will occur by plastic yielding.

Cold-bending Tests.—This is a very useful test of ductility, and can be made at any time in the shop without a special testing machine. The specimen may be bent about a sharp edge in a vice, or to a given radius by clamping an auxiliary plate prepared to this radius with the specimen in the vice. The specimen should be bent slowly by hand, and not struck with the hammer. The objection to this test is the difficulty of standardising it. The International Association for Testing Materials recommends that cold-bending tests should be made by means of a machine, but no special machine has been so far adopted. The specimen may be supported as a beam in a testing machine, and loaded in the centre with a round bar inserted between the specimen and the plate of the machine. After bending in this way through an angle of about 60°, the specimen may be set on end and the bending continued as a column. The specimen may be finally closed upon itself, if sufficiently ductile, by compressing it between the plates of the machine. Several machines have been devised for bending a plate round a definite radius, and there can be no doubt that bending tests of this kind could be made in a much more uniform manner by means of a machine specially arranged for the purpose. Plates cut from portions of structures should be planed on their edges before subjecting them to the bending test.

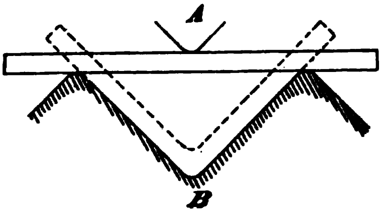

Fig. 16.

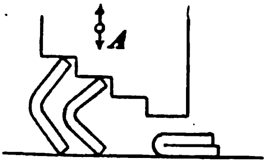

Fig. 17.

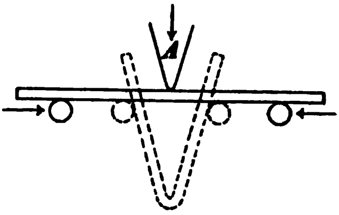

Fig. 18.

Fig. 16 shows a method of bending a plate to a definite angle by forcing the punch A into the block B either by means of a press or steam hammer.

Fig. 17 indicates a reciprocating piece A for making bending tests.

Fig. 18 shows Bauschinger's method of bending a plate about an edge of uniform diameter. The punch A descends and bends the plate; the rollers advance at the same time as indicated by the arrows, so that bending occurs at a uniform rate without much friction.

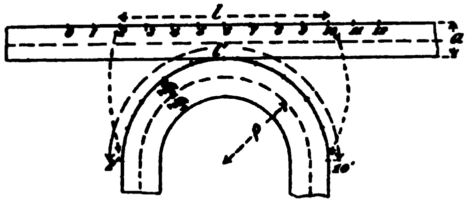

Fig. 19.

Fig. 19 shows a plate bent to a given radius ρ; the thickness of the plate is denoted by a. The extension of the extreme fibre is denoted thus:—

The plate is divided into spaces of about ¼ in. in the centre of the upper surface, and a length l before bending between two divisions, such as 2 and 10, becomes l_1 after bending between the same points; then—

$$\frac{l_1}{l} = \frac{\rho + \frac{a}{2}}{\rho}.$$

$$\text{If } l = 1, \text{ then } l_1 = \frac{\rho + \frac{a}{2}}{\rho},$$

and the extension is $l_1 - l = \dfrac{\rho + \frac{a}{2}}{\rho} - 1 = \dfrac{a}{2\rho}$.

The extension is therefore proportional to the thickness of the plate directly, and inversely proportional to the radius of curvature. The ratio of the thickness of the plate to the radius of curvature is clearly a measure of the ductility of the material as shown by its capacity to undergo bending.

Tetmajer proposed the value of this ratio multiplied by 50 as the "bending-factor" b_f:—

$$b_f = 50\frac{a}{\rho}.$$

$$\text{Since } l_1 - l = \frac{a}{2\rho}; \quad \frac{l_1 - l}{l} = \frac{a}{2l\rho} \quad \therefore \rho = \frac{al}{2(l_1 - l)l}.$$

Martens has pointed out that the actual deformations which take place in bending are not accurately expressed in the foregoing manner, and compares a piece of indiarubber of square section divided into sections shown by parallel dotted lines, and afterwards bent until the ends touched (Fig. 20). An examination of Fig. 20 shows that the occurrences during bending carried to this extreme point are much more complicated than is generally assumed.

The maximum elongation of the extreme fibre in bending tests is probably much greater than that found by means of tension tests.

Bending tests may be made upon bars notched on the tension side with triangular or circular notches, or they may be tested without notches. The breadth of the bar used by Martens is three times the thickness, and the length eighteen times the thickness; again, the radius of the end of the punch (Fig. 18) is made equal to the thickness.

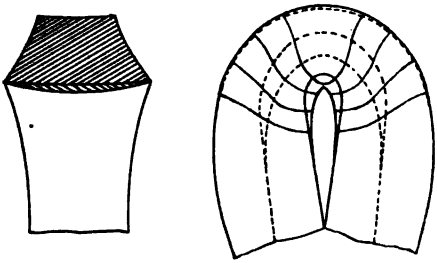

FIG. 20.

The bending test for bars of soft steel used for making rivets consists in heating the bars to a cherry-red, quenching in water at about 80° F., and bending through 180° to a close contact without fracture. Mild steel plates are usually required to bend through 180° round a bar or mandrel having a radius equal to the thickness of the plate without fracture. Medium steel should bend in a similar manner round a mandrel of radius one and a half times the thickness without fracture. Hot-bending tests are frequently made at a red or blue heat, and on annealed and hardened pieces for special purposes. Drifting tests are made by expanding a hole drilled in the test-piece by means of a taper drift. This test may be made at a bright red heat, or cold, and the ratio of the diameters of the hole before and after drifting is taken as the measure of quality. The diameter of the drilled hole should bear a constant ratio to the thickness of the plate in order to obtain comparative results.

Repetitive Tests.—The effect of repeated stress was first shown by Sir W. Fairbairn by means of a riveted girder loaded with about one-third of the load necessary to break it; if applied gradually, the beam failed after 300,000 applications of the load. Afterwards Wöhler made his celebrated experiments on the endurance of iron and steel subjected to repeated stresses, and published his results in 1858. Wöhler's tests, which embrace repeated tension, compression, cross-bending and torsion have been verified and extended to a considerable extent, and a great variety of special machines and apparatus have been designed and used in Great Britain, Europe and America. The names of Martens, Bauschinger, the late Sir B. Baker, Ewing, Unwin, Arnold, and others are associated with investigations of this character, and more recently the National Physical Laboratory, England. Repetitive tests include also impact tests in tension, compression and upsetting, transverse bending, etc. The results of all such tests are expressed by Martens as follows: "Any such stress will produce rupture of the test-piece after a very great number of repetitions thereof only in case the stress invariably applied, hereafter called the initial stress, exceeds a certain amount characteristic of each material".

The limit of this intensity of stress has been called by Launhardt "the factor of resilience".

Wöhler's law is sometimes stated in the following manner:—

"Rupture may be caused, not only by a steady load which exceeds the carrying-strength, but also by repeated application of stresses none of which are equal to the carrying-strength. The difference of these stresses are measures of the disturbance of the continuity, in so far as by their increase the maximum stress which is still necessary for rupture diminishes."

It should be noted that the number of repetitions is smaller the greater the range of stress, not the absolute magnitude of the stress.

Modifications in the Limit of Proportionality and the Yield-point, caused by repeated Stresses.—If a tensile stress is applied to a bar of steel, for instance, which is gradually applied from zero to a stress exceeding the limit of proportionality, the limit will be modified provided that the bar is not in a condition which has been produced artificially. Bauschinger proved that if the stress is released, and the bar immediately reloaded, it will be found that the limit has been raised by the previous straining. If we denote the limit of proportionality and the yield-point by p and y respectively, it has been proved that if a stress is applied greater than that which raised the p limit, the p limit will be further raised up to the point y. Afterwards the p limit decreases and may sink to zero if the point y is greatly exceeded. Ewing has shown[1] the hardening effect of rest after overstraining, by carrying a test to a point a (Fig. 21) beyond the original yield-point, then releasing the load to zero giving the diagram $oyax$. If the load is reapplied immediately and gradually increased as before, a new yield-point y_1 will be produced above the original point. Again, if another specimen be tested similarly to the first and the diagram $oyax$ produced in a similar manner, then instead of applying the stress immediately, an interval of time, such as a day, is allowed before reapplying the stress, it will be found

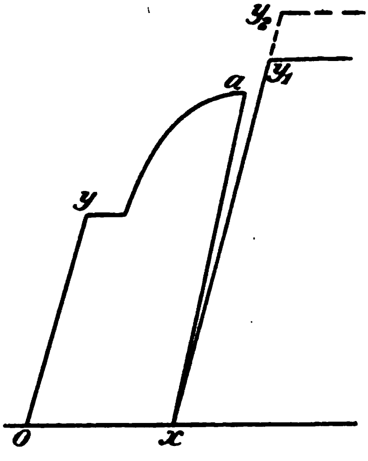

FIG. 21.

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[1] *Proc. Royal Society*, 1880.

that the y point is still further raised to y_2 (Fig. 21), showing that the process of hardening has been going on during the period of rest.

Thurston and Beardslee have shown that if a load exceeding the original p limit be allowed to remain for a definite time without change in intensity, on reapplying the load the p limit and the ultimate strength will be increased, but the elongation will be correspondingly diminished. If the p limit and the y limit have been raised by rest, after release from initial stress greater than that at these points under normal conditions, severe jarring, such as by cold hammering, will again lower the limits p and y. It follows therefore that these artificially raised limits are very unstable. If a bar of steel has been stressed beyond the p limit and retested, it will be found to be very imperfectly elastic, and the strains are not proportional to the stresses producing them, even for small stresses well within the original p limit. If after a few days the bar is again tested it will be found to have partly regained its original elasticity, and the recovery is greater the greater the time interval allowed between the original overstraining and retesting. Muir[1] has shown that if the bar is immersed in a bath of boiling water after overstraining, and retested, it will be found to have completely recovered its elasticity, the strains and stresses being now proportional, but the y limit and ultimate strength will be increased, and the elongation at fracture reduced. By repeated loading and unloading a bar up to its primitive y point, and various raised y points successively, heating up to boiling point between each loading before reloading, it will be possible to raise the y point and ultimate strength to a considerable extent. Muir raised the strength of a piece of steel in this manner from 39 to 49½ tons per sq. in., with a reduction in the elongation of 20 to 12 per cent. Annealing the bar by heating it to redness and allowing it to cool slowly, removes the hardening effect due to strain. Bauschinger found that a bar in which the p limit had been raised may be subjected to several million repetitions of a stress less than this raised p limit, but that the repetition of stresses greater than the maximum p limit attainable must ultimately produce fracture.

It may be generally assumed that repeated stresses exceeding the limit p gradually exhaust the capacity of yielding. Bauschinger[2] found also that when the p limit had been raised in tension, the corresponding limit in compression was lowered, and *vice versâ*: hence if a bar is subjected to stresses alternating between tension and an equal compression, the p limit cannot be raised, and ultimately settles down to its true value. An attempt to raise the p limit in tension lowers it in compression. The true limit p obtained by testing a bar which has been previously subjected to alternating stresses, is always less than the p limit obtained by testing a bar not so treated; and this reduced p limit is approximately equal to the stress for which the bar would endure an infinite number of repetitions of stress alternating between tension and an equal compression.

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[1] *Phil. Trans. Royal Society*, 1899.

[2] This cannot be regarded as absolutely proved, as some experiments by Muir show that raising the tension limit may either lower or raise the compression limit.

The stress under these conditions is about the same in tension and compression, and for mild steel it is about 9½ tons per sq. in.

The following table gives a summary of the most important results obtained by Wöhler:—

TABLE III.

LIMITS OF STRESS FROM WÖHLER'S ENDURANCE TESTS.

Stresses in tons per sq. in. for which fracture occurs only after an indefinite large number of repetitions.

Material.	Opposite stresses.		One stress zero.		Similar stresses.		Statical strength. Range zero.	Constant n for Gerber's Parabola.
	Min.	Max.	Min.	Max.	Min.	Max.		
Phœnix iron . .	− 8·6	+ 8·6	0	15·25	12·0	20·5	22·8	1·33
Krupp's axle steel .	− 14·05	+ 14·05	0	26·50	17·5	37·75	52·0	1·83
Untempered spring steel . . .	− 13·38	+ 13·38	0	25·50	12·5	34·75	57·5	2·14

The following table gives a summary of the most important results obtained by Bauschinger:—

TABLE IV.

BAUSCHINGER'S ENDURANCE TESTS.

Stresses requiring 5 to 10 million repetitions to cause failure.

Material.	Opposite stresses.		One stress zero.		Similar stresses.		Statical strength. Range zero.	Constant n for Gerber's Parabola.
	Min.	Max.	Min.	Max.	Min.	Max.		
Wrought-iron plate .	− 7·15	+ 7·15	0	13·10	11·40	19·20	22·8	1·60
Bar iron . . .	− 7·85	+ 7·85	0	14·40	13·30	22·02	26·6	1·67
Bar iron . . .	− 8·65	+ 8·65	0	15·75	13·20	21·92	26·4	1·53
Bessemer mild steel plate . . .	− 8·55	+ 8·55	0	15·70	14·30	23·80	28·6	1·68
Steel axle . . .	− 10·50	+ 10·50	0	19·70	20·00	32·10	40·0	1·91
Steel rail . . .	− 9·70	+ 9·70	0	18·40	19·50	30·85	39·0	2·00
Mild steel boiler plate	− 8·65	+ 8·65	0	15·80	13·30	22·55	26·6	1·53

The values of the constant n have been deduced by Unwin, and are inserted in the above table for use in the equation of Gerber's Parabola.

Several methods have been proposed which have for their object the representation of the results of the experiments made by Wöhler and Bauschinger, and their extension as far as possible to the various ranges of stress which occur in engineering practice, among which may be noted Gerber's Parabola. If the ranges of stress for which a bar will withstand an unlimited number of repetitions is known for the material under consideration, and are plotted as ordinates, with the corresponding minimum stress as abscissæ, the points fall on a parabolic curve. Professor Unwin expresses these results as follows—

Let f max. and f min. denote the limits of stress, and Δ the range of stress; then—

$$\Delta = f\text{ max.} \mp f\text{ min.}$$

The upper sign is to be taken when the stresses are of like kind, and the lower sign when they are of opposite kind, as in alternating stresses.

Let f denote the statical breaking stress; then the equation to Gerber's Parabola is—

$$\left(f_{\text{min.}} + \frac{\Delta}{2}\right)^2 + n\Delta = f^2.$$

If the statical strength f be known, and the value of f min. and f max. for any range of stress at which the bar stands a practically unlimited number of repetitions before breaking, then n can be determined, and the limits of stress for all conditions of loading can be calculated. The parabolas are drawn from this equation, using the results recorded in Tables III. and IV. Fig. 22.

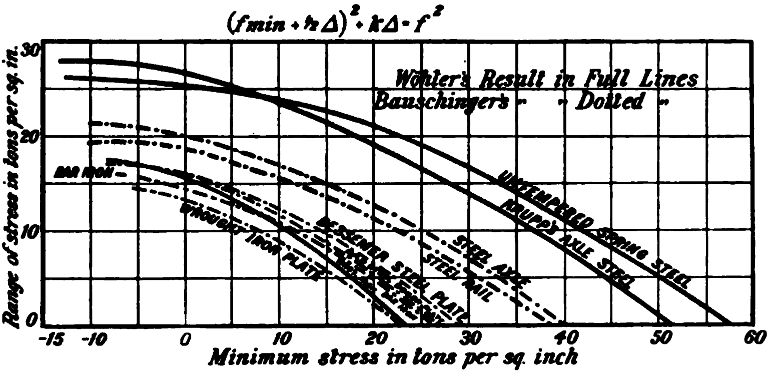

FIG. 22.

Gerber's Parabola is an empirical representation of the results discussed, but it fits these results very fairly. The equations of Launhardt and Weyrauch are as follows :—

Let b denote the ultimate breaking strength after an infinite number of repetitions; s, the statical strength as found by the testing machine when the load is gradually applied; p, the primitive strength, when the stress is entirely removed, before being reapplied it corresponds fairly well with the yield-point; v, the vibrating strength, obtained by an indefinite number of repetitions of stresses alternating between tension and an equal compression.

Then Launhardt's equation for stresses of one kind is—

$$b = p + (s - p)\frac{\text{min. stress}}{\text{max. stress}}.$$

Weyrauch's equation for stresses of opposite kinds is—

$$b = p - (p - v)\frac{\text{max. } x}{\text{max. } y},$$

where max. x denotes the smaller of the two values. Roughly, the values of v, p, and s are as $1 : 2 : 3$. If we plot the results obtained by means of these formulæ on the same system as that adopted by Gerber, the curves obtained will also be parabolic. We may represent the results of Wöhler's and Bauschinger's endurance tests in a more simple manner by means of a single equation, if we put $s = \frac{3}{2}p = 3v$ and consider like stresses plus, and opposite stresses minus, thus:—

$$b = p\left(1 + \tfrac{1}{2}\,\frac{\text{min. stress}}{\text{max. stress}}\right).$$

The effect of repeated stresses on the metal is purely local, and confined to the plane of fracture only. The examination of highly polished sections of iron and steel by means of a microscope magnifying from 200 to 500 times, has revealed the existence of numerous defects termed micro-flaws.[1] The micro-flaws are extended and developed by repeated stresses in an irregular manner in the cross-section which ultimately becomes the plane of fracture, and as might be anticipated the results of tests are very irregular. It has been assumed that the failure is due to the dynamic effect[2] rather than to the repeated effect. The varying loads may be considered as equivalent to suddenly applied loads on this theory, and a piece of material will fracture when the momentary stress due to sudden application exceeds the statical strength. This view was developed by Johnson,[3] who obtained an expression for safe stresses in material. Prof. Goodman[4] has constructed a more simple diagram (Fig. 23), showing the results of repeated stresses exceeding four millions. The horizontal scale is immaterial; the vertical scale shows the ratio of the applied stress to the static breaking stress. The minimum stress is plotted on the line *aob*, and the corresponding maximum stress, which may be repeated

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[1] J. Andrews, F.R.S., M.Inst.C.E., *Engineering*, July, 1906.

[2] Fidler's *Practical Treatise on Bridge Construction.*

[3] Johnson's *Materials of Construction.*

[4] Goodman's *Mechanics Applied to Engineering.*

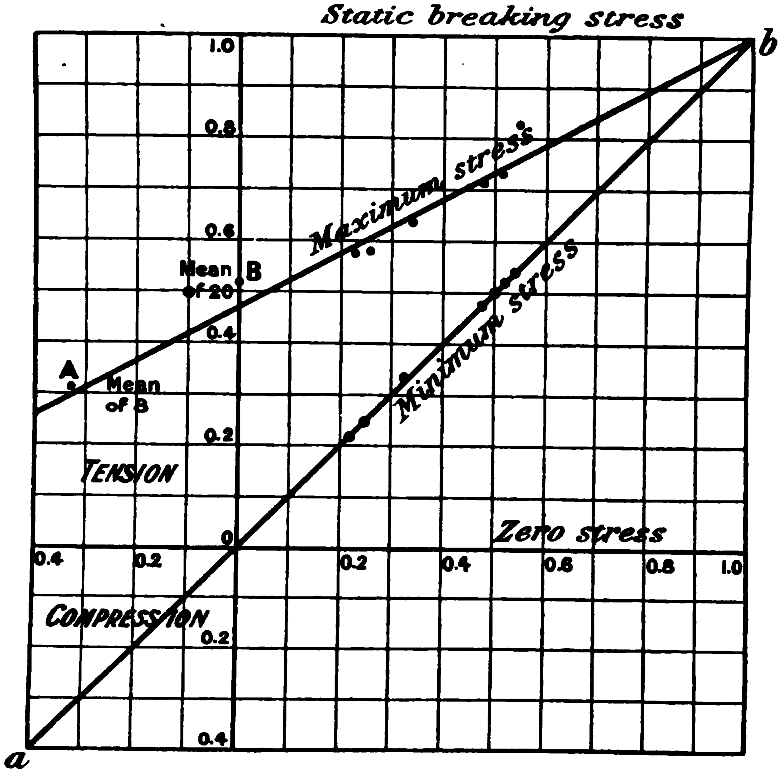

FIG. 23.

four million times, is shown in dots above. According to the dynamic theory, all the points would lie on the line marked max. stress, thus :—

$$zy + 2xy = mb \text{ (Fig. 23),}$$

or, min. stress + 2(max. stress - min. stress) = statical stress.

The results are as nearly on this line as could possibly be expected from experiments of this character.

Impact Tests.—It has long been known that ordinary tension tests, in which the load is gradually applied, do not always reveal the capacity of the material to withstand loads suddenly applied, but that this property is best determined by subjecting the material to shock, in a suitable machine under standard conditions.

Impact tests are becoming very important, although so far they are not specified in connection with material required for structures, but they are used for railway material, such as rails, tyres, axles, and also in the material used for engines and machinery, since in such cases the stresses developed under normal conditions are always more or less suddenly applied. Impact machines[1] have recently been much improved in design and construction, and impact tests have received considerable attention. The machines devised by Charpy, Guillery and Frémont are very satisfactory in recording the energy required to rupture a test-piece with a single blow.

Impact tests consist of: upsetting or bulging tests, transverse tests on plain or notched bars, and tension tests. The total energy of a number of blows required to produce fracture may be used, or the energy of a single blow, and the result expressed per unit of volume of the test-piece.

If Wh denote the energy in foot-pounds, and v the volume of the test-piece in cub. in., then—

$$\mathrm{I} = \frac{\mathrm{W}h}{v} = \text{specific impact.}$$

Upsetting or Bulging Tests.—For these tests a prismatic or cylindrical body is used of length l and diameter d, such that—

$$l = d, \text{ or } l = 0{\cdot}886d; \text{ i.e., } \frac{\sqrt{a}}{l} = 1.$$

The upsetting will be:—

$$\frac{-\lambda}{l} = \frac{l - l_1}{l}, \text{ or in per cent.} = \left(1 - \frac{l_1}{l}\right)100 \text{ (Fig. 24).}$$

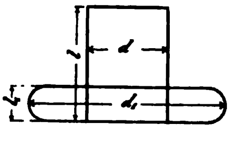

FIG. 24.

If the volume does not change during deformation there will be a relation between d and l_1 which, neglecting bulging, is—

$$v = v_1, \text{ or } \frac{\pi}{4}d^2l = \frac{\pi}{4}l_1d_1^2,$$

or when $d = l = 1$—

$$l_1 = \frac{1}{d_1^2}.$$

It has been found by Kick that the velocity of impact exerts a small influence on the deformation of lead. Generally it has been found that heavy blows produce greater deformation than light blows of equal specific impact. Martens has proposed the term "Crushing Factor" for the specific impact in foot-pounds which, when applied to a body of fundamental shape (sphere, cube, standard-plug) in a single blow, will either just produce rupture or a crushing deformation = 80 per cent.

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[1] Impact machines should be built in accordance with the recommendation of the International Society for Testing Materials as described in the fourth Congress Report, 3rd-6th Sept., 1906.

Impact Transverse Test.—Fig. 25. The impact transverse test, commonly known as the drop test, is made by placing a bar across two supports, and dropping a hammer or ram so as to strike it suddenly in the centre. The deflection v may be measured by means of a straight edge of length l touching at d, d; the straight edge is provided with a slide at the centre. The deflection V_1 may be measured with three-legged compasses. The drop test is largely used for testing railway materials such as axles, tyres, and rails. In the British Standard Specifications[1] it is required, in the case of carriage and wagon axles, to place the axle upon bearings resting on a block of not less than 5 tons weight, supported on a rigid concrete or other solid foundation, and subjected to five blows of a ton falling from a height given in the following table. The axle must be turned after the first and third blow, and shall not be broken in the centre after testing.

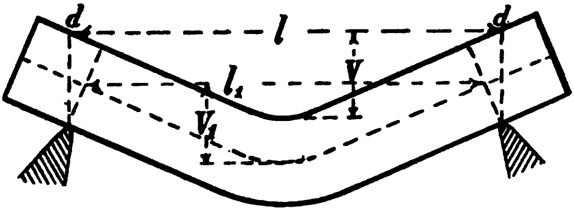

Fig. 25.

TABLE V.

Diameter of axle at centre.	No. of blows.	Height of fall in feet.	Distance apart of bearings.	
			Feet.	Inches.
Under 3¾ in.	5	15	3	0
3¾ in., and under 4 in. . .	5	16	3	0
4¼ in., and under 4½ in. .	5	20	3	6
4½ in., and under 4¾ in. .	5	20	3	6
5 in., and under 5¼ in. . .	5	24	4	0
5¼ in., and under 5½ in. .	5	28	4	0
5½ in., and under 5¾ in. .	5	28	4	6
5¾ in., and under 6 in. . .	5	32	4	6
6 in., and over . . .	5	35	5	0

Transverse Tests on Notched Bars.—It is well known that certain steels which give normal results when tested in tension, tempering and bending, may appear, on the contrary, extremely brittle if tested by impact using a bar notched on its under side by means of a saw, milling cutter, or other device for producing a notch of definite size and shape. Impact tests have recently been introduced for the determination of brittleness using specimens of the form and dimensions shown in Figs. 26 and 27. Fig. 27 may be used with the special machines devised for this purpose, such as by Guillery, Charpy and Frémont. Fig. 26 requires the large model of the Charpy type.

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[1] British Standard Specifications for Railway Rolling Stock, Bull-headed and Flat-bottom Rails.

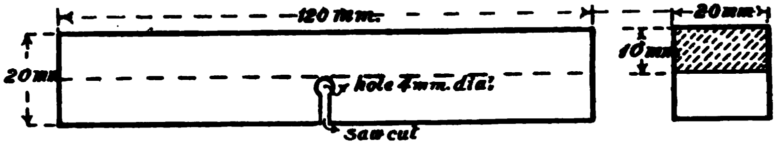

Fig. 26.

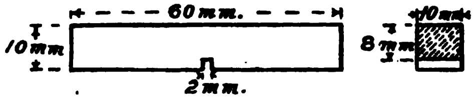

Fig. 27.

Static bending tests on notched bars have also been used to determine the degree of brittleness by means of the total energy necessary to produce rupture, but the results are less decided than with impact tests. Impact tests on notched bars are usually made in a single blow, by means of the machines referred to, which are arranged to show the exact energy absorbed by the test-piece in kilogramme meters. This energy is expressed as per square millimeter of the cross-section through the notch.

Mesnager[1] has investigated this subject on various kinds of steel; the results of some of these are given in the table on next page.

If a specimen of soft steel is notched on its under side, and polished so as to show the distribution of the strain lines in the depth, then subjected to partial bending, the strain lines consist of the interposition of two elementary deformations, one of swelling, the other of a depression (Figs. 28 and 29). In Fig. 28 the depression EFGBH is nearly an ellipse; the swelling is a portion of the ellipse having the same

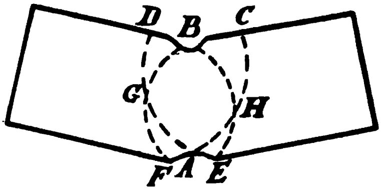

Fig. 28.

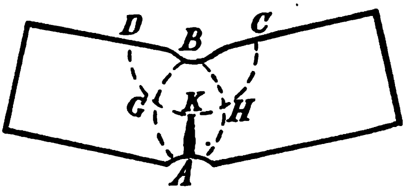

Fig. 29.

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[1] M. Mesnager, Director of the Laboratory de l'Ecole de Ponts et Chaussées, Paris, Brussels Congress Int. Assoc. for Testing Materials,

major axis. The two ellipses have the part AGBH in common, and thus the two deformations are in part neutralised. Rupture takes place along the synclinal GF and HE.

TABLE VI.

GIVING COMPARATIVE RESULTS OF DIFFERENT TESTS.

Tension test.		Static bending.		Bending by shock.			
				Notched by saw.		Notched by cutter.	
Tensile strength. Kg. per sq. mm.	Elongation in 20 cm.	Kg. m. along.	Kg. m. across.	Kg. m. along.	Kg. m. across.	Kg. m. along.	Kg. m. across.
47	24	9·5	8·0	13·0	6·0	10·0	6·5
60	25	4·5	2·5	4·5	4·0	6·0	6·0
44	28	12·5	10·0	6·0	6·0	6·5	4·5
37	30	2·2	20·0	9·0	7·0	1·0	7·5
33	25	12·5	2·0	9·0	3·0	3·0	3·2
33	32	21·0	20·0	5·5	3·0	4·0	3·5
45	27	5·5	6·0	3·0	4·0	5·5	4·7
60	22	15·5	4·0	7·0	3·5	6·7	3·0
50	20	5·5	12·0	7·0	5·0	6·0	4·0
40	23	22·5	17·5	22·0	13·0	21·0	19·0
53	23	4·0	3·0	5·0	2·5	5·2	3·0
49	27	20·5	6·0	21·5	5·0	24·0	4·0
46	27	25·0	12·0	26·0	18·5	30·0	17·0
53	28	22·0	10·0	30·0	21·5	24·0	23·5
50	26	24·5	6·5	23·0	5·0	21·5	4·5

In fragile steel (Fig. 29) the ellipse caused by the compression or swelling is reduced to nearly nothing. The rupture is made downwards by tension following the line AK (Fig. 29), and is effected by a small expenditure of work. If the point K is near A, the metal will not be weak; if it goes towards B the metal will be weaker the nearer it is to B. Thus the prominence of the swelling and the position of the point K are important factors in this determination of the fragility of the specimen.

Mesnager has shown by means of polarised light the distribution of the stresses about the region of the notch in glass bars. The tension developed on the edges of the hole in a round notch may be double that in a similar bar of the same material without a notch; with other shapes the tension varies with the shape and depth of the notch.

Mesnager's conclusions on the results of his investigations on notched bars subjected to impact are as follows:—

(1) Generally speaking, the results vary less in the case of large test-pieces with cylindrical notches, than in small.

(2) The relationship between the number of kilogrammeters per

square centimeter of the section broken and the angle of deformation expressed in degrees is given by the following equations :—

Small test-pieces, $8 \times 10 \times 30$, $K = 0{\cdot}375D$.

Large test-pieces, $20 \times 20 \times 120$, $K = 1 + 0{\cdot}58D$.

(3) Hence a measurement of the angle of rupture may be substituted for that of the work absorbed in rupture.

(4) The relationship between the tensile strength expressed as kilogrammes per square centimeter, the angle of rupture, and the number of kilogrammeters per square centimeter, are given by the following approximate equations :—

Small pieces, $R + 2{\cdot}66D = 95$; $R + 7{\cdot}1K = 95$.

Large pieces, $R' + 1{\cdot}72D' = 87$; $R' + 3K' = 90$.

(5) A porous material appears to absorb more energy in rupture than one that is sound.

(6) Among homogeneous metals the large test-pieces yield remarkably regular results.

(7) The large test-pieces give the same results whether the notch is made with a drill or with a milling cutter.

(8) Variations of 5 mm. in the distance between the supports, and in the thickness of the large test-pieces, only affect results to a small extent, provided the section to be broken remains the same.

(9) The width of notch, on the other hand, exerts a great influence on the results.

The object of the notch in Fig. 26 introduced by Charpy is to ensure its exact reproduction in the test-pieces, using a twist drill and afterwards cutting away the lower portion. The deeper and narrower the slot the greater will be its influence on the result. It has been found that worn saws and new saws give results which are sensibly the same. The following table gives a few of the author's results obtained with bars 60 mm. × 10 mm. × 10 mm. notched on the underside with a saw 2 mm. wide × 2 mm. deep.

TABLE VII.

Description.	Energy of rupture kg. per sq. cm.	Angle of rupture in degrees.	Remarks.
Bessemer steel . .	27·8	44	Guillery machine
Axle steel . . .	16·2	—	Charpy „
Tyre steel . . .	6·9	173·9	Guillery „
Steel in bore casings .	15·2 to 36·85	32·5 to 21·8	„ „
Nickel steel 6 % . .	21·2	158	„ „
„ „ 3 % . .	15·9	174	„ „

Impact Tension Tests.—The great difficulty in these tests with the ordinary drop hammer consists in the determination of the actual energy absorbed by the test-piece, in consequence of the shackles used for holding the test-piece, which receive the blow and transmit it to the test-piece. The writer has obtained regular results using a number of blows, about ten, both in the total energy necessary to produce fracture, and in the elongation produced by each blow; but without special apparatus it is not possible to determine the energy of a single blow which will just cause fracture except by using a large number of test-pieces.[1] With pendulum hammer machines such as Charpy's, it is not only possible to determine the energy of a single blow producing fracture, but the arrangements allow the blow to be delivered to the test-piece in a more direct manner, thus avoiding the loss due to the shackles in the ordinary drop machine. In all impact tests it is necessary to use identical apparatus sections and dimensions under identical conditions in order to obtain results which are strictly comparable. It is best to use test-pieces of standard forms such that—

$$l = 11{\cdot}3\sqrt{a},$$

and to calculate the specific impact on the volume of the parallel portion of the test-piece in cubic inches.

HARDNESS, TOUGHNESS AND BRITTLENESS.

Hardness.—The hardness of a material is difficult to define in an exact manner; we may compare the resistance of a piece of metal to the penetration of a punch of definite shape, and note the volume of the material displaced; or we may substitute a specific impact for the load or pressure applied gradually. Again, we may compare the resistance to scratching by means of a conical point, by determining the load necessary to produce a scratch of a definite width after the method devised by Martens. The tensile strength, ductility, the plastic yielding, as shown in torsion and bending tests, are all more or less measures of hardness. None of these methods furnish an absolute measure of hardness, but they all can be used to give relative measures of hardness in regard to the particular method adopted. Static penetration was used by Messrs. Calvert and Johnson by means of a punch of the form of a truncated cone. Middleburgh used a special form of punch in which the material displaced was proportional to the length of the impression. Unwin used a short bar of steel as an indenting tool under a gradually applied pressure. Kirsch used a punch not unlike Messrs. Calvert and Johnson, but he measured the deformation and determined the limits of elasticity and fluidity. Kerpley used a conical punch, while Professors Rudeloff and Föppl used cylinders crossed at right angles to each other in order to compare the hardness by means of the impression produced when the cylinders were pressed together. Keep and Rodman each used a special form of punch in an impact testing machine. Recently a much more satisfactory method has been introduced by Brinell, which is of sufficient importance to describe more fully. Brinell's method of determining the hardness of metals consists in using a small ball of tempered steel 10 mm. in diameter pressed gradually into the metal specimen with a force of 3000 kg. for iron and steel and 500 kg. for the softer metals, such as copper, zinc, tin, and their alloys. The pressure is maintained for two minutes, and is divided by the area of the surface of the impression made in the specimen. The quotient gives the hardness expressed in kilogrammes per square millimeter. The Brinell apparatus is of simple character and takes various forms, but the one used by the author is shown in Fig. 30.

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[1] The energy absorbed by the test-piece during fracture by a single blow in all drop tests may be calculated from the diagram recorded on a revolving drum provided with a tuning-fork giving time, as used by Prof. Hatt in timber tests.

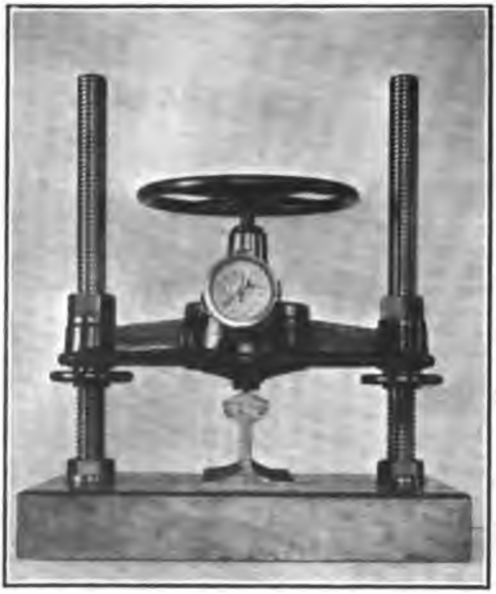

FIG. 30.

It consists of a specially designed hydraulic press operated by means of a hand wheel and screw. Two pressure gauges indicate the total pressure applied, which should be periodically calibrated. The illustration (Fig. 30) shows a rail subjected to the indentation of the steel ball.

Since the surface of the impression is proportional to the square of the diameter, tables have been prepared to facilitate calculations, giving the hardness when the diameter of the impression has been accurately measured to $\frac{1}{100}$ of a millimeter. This is done with a measuring microscope, or by means of a special form of micrometer, or by means of the glass scale used by Le Chatelier, which gives the diameter to $\frac{1}{10}$ of a millimeter.

This apparatus has been used to determine the tensile strength, yield-point and ductility, but the author has only used it to determine the hardness, and to compare the actual tensile strength with the result given with the Brinell apparatus.

The following table is useful for the determination of Brinell's hardness number:—

TABLE VIII.

TABLE FOR THE DETERMINATION OF HARDNESS BY BRINELL'S METHOD.

Diameter of the impression, mm.	Hardness number for the pressure, in kg.		Diameter of the impression, mm.	Hardness number for the pressure, in kg.	
	3000.	500.		3000.	500.
2·00	946	158	4·50	179	29·7
2·05	898	150	4·55	174	29·1
2·10	857	143	4·60	170	28·4
2·15	817	136	4·65	166	27·8
2·20	782	130	4·70	163	27·2
2·25	744	124	4·75	159	26·5
2·30	713	119	4·80	156	25·9
2·35	683	114	4·85	153	25·4
2·40	652	109	4·90	149	24·9
2·45	627	105	4·95	146	24·4
2·50	600	100			
2·55	578	96	5·00	143	23·8
2·60	555	93	5·05	140	23·3
2·65	532	89	5·10	137	22·8
2·70	512	86	5·15	134	22·3
2·75	495	83	5·20	131	21·8
2·80	477	80	5·25	128	21·5
2·85	460	77	5·30	126	21·0
2·90	444	74	5·35	124	20·6
2·95	430	73	5·40	121	20·1
			5·45	118	19·7
3·00	418	70	5·50	116	19·3
3·05	402	67	5·55	114	19·0
3·10	387	65	5·60	112	18·6
3·15	375	63	5·65	109	18·2
3·20	364	61	5·70	107	17·8
3·25	351	59	5·75	105	17·5
3·30	340	57	5·80	103	17·2
3·35	332	55	5·85	101	16·9
3·40	321	54	5·90	99	16·6
3·45	311	52	5·95	97	16·2
3·50	302	50			
3·55	293	49	6·00	95	15·9
3·60	286	48	6·05	94	15·6
3·65	277	46	6·10	92	15·3
3·70	269	45	6·15	90	15·1
3·75	262	44	6·20	89	14·8
3·80	255	43	6·25	87	14·5
3·85	248	41	6·30	86	14·3
3·90	241	40	6·35	84	14·0
3·95	235	39	6·40	82	13·8
			6·45	81	13·5
4·00	228	38	6·50	80	13·3
4·05	223	37	6·55	79	13·1
4·10	217	36	6·60	77	12·8
4·15	212	35	6·65	76	12·6
4·20	207	34·5	6·70	74	12·4
4·25	202	33·6	6·75	73	12·2
4·30	196	32·6	6·80	71·5	11·9
4·35	192	32·0	6·85	70	11·7
4·40	187	31·2	6·90	69	11·5
4·45	183	30·4	6·95	68	11·3

The author has tested various specimens of steel by the Brinell method, and compared the results with those obtained by the ordinary tension test. The value obtained by dividing the actual tensile strength by the Brinell hardness number is fairly constant for the same qualities of steel.

For eight tests of structural steel the results are shown in the following table:—

TABLE IX.

	Tensile strength tons per sq. in.	Brinell's hardness number.	Ratio of the tensile strength to hardness number.
	28·9	170	·170
	30·2	149	·203
	25·3	141	·180
	25·6	140	·183
	27·0	140	·193
	25·8	140	·184
	24·9	137	·182
	25·6	134	·191
Mean	26·7	143·9	·186

In the case of axle steel the results are given in the following table:—

TABLE X.

	Tensile strength tons per sq. in.	Brinell's hardness number.	Ratio of the tensile strength to hardness number.
	34·0	167	·202
	36·5	168	·216
	33·95	205	·165
Mean	34·8	180	·194

Tyre steel and crucible steel give different ratios corresponding with their tensile strengths. If we apply Brinell's method to determine both the hardness number and the tensile strength, we may select the value of this ratio for structural steel, 0·186, which would give a tensile strength of 26·7 tons per sq. in.; the degree of approximation is seen by comparing the mean value with any of the eight values given. For more complete information on this test the student is referred to Brinell's paper.[1]

Scoring Hardness.—Professor Martens has devised a special sclerometer for the determination of the scoring hardness, and has developed the method first applied by Turner. A diamond with a conical point is carried at the end of a lever which can be loaded by means of a poise weight moved along a graduated arm. The specimen is moved under the diamond point, and the mean width of five scratches measured by means of a microscope. The process is repeated for five different loads and the results plotted. From this diagram the load is found which would produce a scratch having a width of $10\mu = 0{\cdot}01$ mm.

Toughness and Brittleness.—Toughness and brittleness are difficult to define, as there is no absolute measure of these qualities. Pitch is tough when deformed slowly, and brittle when broken by impact; glass is brittle at ordinary temperatures, but becomes tough when heated. Various methods have been proposed for the determination of toughness. Martens suggests that tension tests should be used to determine the ultimate strength σ_t, and the yield-point σ_y, also the percentage of elongation ϵ per cent.; then the toughness T is found thus:—

$$T = \frac{\sigma_t\ \epsilon \text{ per cent.}}{\sigma_y\, 100}.$$

If the ultimate strength of steel is 28 tons per sq. in., the yield-point 16·8 tons per sq. in., and the elongation 25 per cent., then—

$$T = \frac{28 \times 25}{16{\cdot}8 \times 100} = 0{\cdot}416.$$

Plasticity.—Soft inelastic tough materials are usually plastic, and can be subjected to great changes of shape without fracture.

Martens proposes as a measure of plasticity the ratio of the toughness to the yield-point multiplied by 10—

$$P = \frac{T}{\sigma_y} \times 10 = \frac{0{\cdot}416}{16{\cdot}8} \times 10 = 0{\cdot}247,$$

for the steel giving a toughness of 0·416.

CAST-IRON.

The application of cast-iron in engineering construction on a fairly extensive scale dates from the beginning of the nineteenth century. In 1776-79 Abraham Darby built the first cast-iron bridge at Coalbrook Dale on the Severn. The bridge was constructed of cast-iron ribs, and stood until 1902, when it partially gave way; it was, however, repaired under Sir B. Baker, who considered it good enough for another 100 years. For many purposes cast-iron will probably never be entirely superseded by any other material; it consists of metallic iron together with about 7 per cent. or more of the total weight of carbon, silicon, phosphorus, sulphur, and manganese. Cast-iron melts at about 2700° F., and when cold is more or less hard and brittle. In the fused condition it consists of carbon in an iron solution, more or less saturated, usually containing 1·5 to 5 per cent. of carbon. Cast-iron is produced by smelting fragments of iron ore mixed with fuel and limestone flux in a blast furnace; the resulting product run into moulds is known as "pig-iron". Pig-iron is rarely, if ever, used for anything except to be remelted into cast-iron or for the manufacture of wrought-iron and steel. Cast-iron has the advantage that it may be made to take any desired form; a pattern is made of timber or other material, a mould is made in the foundry sand, the pattern is removed and the metal poured in. Cast-iron resists corrosion better than wrought-iron or steel: it possesses a high compressive and comparatively low tensile strength; it cannot be forged or welded; it breaks short without elongation, and offers slight resistance to suddenly applied loads. Castings necessarily have initial stresses from inequalities in the rate of cooling which may be removed by annealing; they are also liable to hidden defects which render their strength uncertain. Cast-iron may be roughly divided into two classes, white and grey. White cast-iron contains the carbon in solution, which does not separate out on cooling; whereas in grey cast-iron only a portion of the carbon is in solution, the larger part separates out on cooling as graphite mechanically distributed through the mass producing the grey appearance. Grey cast-iron is divided into three classes; it is less hard and brittle than white, and is used for ordinary castings. The physical properties of cast-iron are influenced by the various elements associated with it: silicon, 0·5 to 5 per cent., tends to increase the quantity of combined carbon; about 1 per cent. of aluminium changes white cast-iron into the grey variety. Manganese should not exceed 1 per cent.; sulphur, 0·15 per cent.; phosphorus, 0·2 to 0·6 per cent. Phosphorus produces fluidity. Remelting cast-iron is generally done in a small blast-furnace known as a "cupola," with coke, charcoal, or gas fuel. The density of cast-iron varies from 6·80 to 7·60, the white being denser than the grey.

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[1] "The Brinell Hardness Test and its Practical Applications," by J. A. Brinell and Gunnar Dillner, Int. Assoc. Testing Materials, Brussels Congress, 1906.

Mechanical Properties.—There is no well-defined limit of elasticity or coefficient of elasticity in cast-iron. The ordinary qualities have an apparent coefficient of from 12,000,000 to 14,000,000 pounds per sq. in.

The tensile strength is shown in the following table:—

TABLE XI.

Description.	Tensile strength tons per sq. in.	Specific gravity.
Good pig-iron . . .	8·9	7·10
Tough cast-iron . . .	11·2	7·22
Hard cast-iron . . .	13·4	7·28
Good tough gun-iron . .	13·4	7·25

Hardness is best determined by Martens' sclerometer so far as it relates to cutting by machine tools.

The strength of average castings is about 9 tons per sq. in. The safe working stresses for water-pipes and similar purposes is 3000 to 3500 pounds per sq. in.

The crushing strength of castings is greatest when the specific gravity is slightly greater than that producing the highest results in tension. For ordinary castings the compressive strength varies from 38 to 48 tons per sq. in., but with a specific gravity of 7·27 it should be about 70 tons per sq. in. The transverse strength expressed by the formula—

$$f = \frac{3Wl}{2bd^2},$$

is about 40,000 pounds per sq. in. for ordinary castings if determined on the ordinary British Standard bars. The tests for tensile strength should be made on standard bars prepared in a similar manner to steel bars, in preference to rough bars cast in a form suitable for testing. Compression tests are best made on turned cylinders 1 in. diameter, and 2½ in. long, which allows the shortening to be determined with Martens' mirror extensometers. Transverse tests are made in Great Britain with bars 39 in. long, 1 in. wide, and 2 in. deep supported on bearings 36 in. centres, and loaded in the centre. A load of 3000 pounds is specified for ordinary castings which gives the modulus of rupture f, 40,500 pounds per sq. in.

The average deflection is given by the following equation:—

$$v = \cdot00000025\frac{l^2}{d}f.$$

This gives for ordinary castings, of the dimensions above described, a deflection of 0·324 in.

In America it is usual to specify bars 1 in. square on 12 in. centres, but the modulus of rupture will not be the same as that found for the same material on British Standard bars, since they are not geometrically similar and the grain changes by slow cooling.

The shearing strength of cast-iron has not been much investigated, and the results obtained by different authorities vary, owing largely to the difficulty of making such tests accurately. Messrs. Platt and Hayward obtained with cast-iron having a tensile strength of 11·4 tons per sq. in., and grooved test-pieces 0·75 in. diameter, a shearing strength of 5 tons per sq. in., whereas Mr. Izod obtained about 14 tons per sq. in. with similar material, and with iron having a tensile strength of 13·4 tons per sq. in., the shearing strength was 17·4 tons per sq. in. The torsional strength σ, found from the ordinary formula—

$$\sigma = \frac{T}{0{\cdot}196d^3},$$

where T denotes the twisting moment, and d the diameter of the test-piece, is about 14 tons per sq. in., but in Platt and Hayward's experiments it varied from 8·25 to 23·5 tons per sq. in. for different irons. The torsional strength is not the shearing stress at the extreme fibre, no more so than the modulus of rupture is the extreme fibre stress in transverse tests. The coefficient of rigidity C—

$$C = \frac{32T}{\pi i d^4},$$

where i denotes the angle of torsion per unit of length, varied from 2947 to 3402 tons per sq. in. in Platt and Hayward's tests. Cast-iron will not stand repetitive tests or impact tests, like wrought-iron or steel.

WROUGHT-IRON.

Wrought-iron consists of about 99 per cent. of pure iron intimately associated with slag; it melts only at a very high temperature, and is capable of being forged and welded. Wrought-iron is produced from a mixture of pig-iron and scrap in a reverberatory furnace by the process known as puddling; its well-known fibrous quality is developed by squeezing and rolling. The fracture of good wrought-iron in tension tests is fibrous with generally a small percentage of crystal. All wrought-iron fractured by impact shows a crystalline fracture, but it does not become crystalline by repeated stresses or by shock. The crystalline appearance of the fracture is due to the fibres being suddenly broken before they have had time to draw out and produce the fibrous fracture.

The presence of sulphur in excess of 0·1 per cent. produces brittleness at a red heat, and of phosphorus produces brittleness at ordinary temperatures. The most valuable quality of wrought-iron is its capacity for being welded, and it is used in chains, hooks, railway couplings, and wherever welding is necessary, in preference to mild steel, but for nearly every other purpose it has been superseded by structural steel. The strength of a weld is rarely equal to that of the solid bar; in iron chain cables welds of ½ to 2½ in. gave a strength of 83 per cent. of that of the iron bars as an average of 216 tests, but the minimum strength may probably fall as low as 50 per cent.

In regard to the results of tests of wrought-iron, these will be only very briefly referred to, for the reason already stated.

Three well-known brands of wrought-iron plate known as B, BB, and BBB Staffordshire were formerly much used, and they may be taken as representing the qualities of structural iron. The results of tests show that the tensile strength of the B quality was 21 tons per sq. in. with the fibre, and 18 tons per sq. in. across it, whereas the contractions of area along and across the fibre were 10 per cent. and 5 per cent. respectively. In the BBB quality the tensile strength along and across the fibre was 22 and 19 tons per sq. in. respectively, and the contraction of area 17 per cent. and 7½ per cent. respectively. The difference in the quality is shown more by the ductility than the tensile strength. In the best Yorkshire plate, which is superior to BBB Staffordshire, the strength is 22 tons per sq. in. along, and 20 tons per sq. in. across the fibre, the contraction of area along and across the fibre being 20 and 12 per cent. respectively, provided the plates were at least 3 ft. wide. The approach to equality in strength and ductility along and across the fibre indicates the superiority of the Yorkshire over the Staffordshire BBB plates. Narrow widths of plates give a much greater difference than wider plates along and across the fibre. Yorkshire iron is used for rivets, chain cables, and best smith's work, and gives excellent results in impact tests.

STEEL.

Steel is a material which may be defined as consisting mainly of iron which holds in solution, in whole or in part, certain other essential and non-essential elements. The essential elements are carbon and manganese, which render the steel capable of fulfilling practical requirements. The non-essential elements in steel, which it invariably gets from the present process of manufacture, are: silicon, sulphur, phosphorus, and probably oxygen, hydrogen, and nitrogen. The relative proportions of these elements may be varied, and by special care one or two of them eliminated without altering the characteristic properties of the steel. Three kinds of steel are in common use to-day, known as—

Crucible Steel.
Bessemer Steel.
Open Hearth Steel.

Open hearth steel consists of two varieties, known as acid and basic, depending on the lining of the furnace with acid fire-proof material such as sand or silica, or basic material such as dolomitic lime stone, with calcined lime added with the charge. In the acid process the ore must be comparatively free from sulphur and phosphorus; in the basic processes these substances are eliminated, and hence cheaper ores can be used. Open hearth steel is largely used in structures of all kinds, almost entirely superseding wrought-iron, and this material is known as "structural steel".

Structural steel consists broadly of three varieties, mild, medium, and hard steel (sometimes called high steel), depending mainly on the percentage of carbon contained. Each variety, however, merges into the other thus: the mild variety contains from 0·1 to 0·2 per cent. of carbon, the medium from 0·2 to 0·3, and the high steel from 0·3 to 0·45 per cent. of carbon. In American specifications it is usual to speak of rivet, soft, and medium steel, the high steel being very rarely used. Rivet steel must have a tensile strength of from 48,000 to 58,000 pounds per sq. in., and an elongation of 26 per cent. on a gauged length of 8 in. It must also bend through an angle of 180°, and then the two portions should be brought in close contact without sign of fracture. Soft steel must have a tensile strength of 52,000 to 62,000 pounds per sq. in., an elongation of 25 per cent. on 8 in., and should also bend flat on itself without fracture. Medium steel must have a tensile strength of 60,000 to 70,000 pounds per sq. in., an elongation of 22 per cent. in 8 in., and should bend through 180° to a diameter equal to the thickness of the piece tested without fracture.

In the above tests it is usually specified that the elastic limit (apparent elastic limit is understood) should not be less in any case than one-half the ultimate strength.

The test-pieces should have the sizes and proportions recommended by the British Standards Committee for structural steel test-pieces A and B [1]; or the tests should be made in accordance with the instructions issued by the International Society for Testing Materials.[2]

The structure of all steel, as revealed by means of the microscope, and the physical properties, as shown by mechanical tests of various kinds, appear to depend chiefly on—

(1) The ultimate chemical composition.

(2) The mechanical treatment it has undergone.

(3) The conditions under which it has been heated or cooled, that is to say its "heat treatment," which may induce the ultimate components of the mass to regroup themselves in new combinations, thus causing one set of minerals to be replaced by another.

The microscopic analysis of iron and steel was inaugurated by

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[1] Forms of Standard Tensile Test Pieces.

[2] *Methods of Testing Metals and Alloys*, Brussels, 3rd-6th Sept., 1906, published by Spon Ltd., London.

Dr. Sorby, who demonstrated that metals should be regarded as crystallised igneous rocks, and this view is now held by all scientific and practical metallurgists. It is necessary to anneal eye-bars of bridges, and other numbers which have been subjected to forging in order to remove internal stresses. Annealing consists in reheating the steel to about 1150° F. or higher, and allowing it to cool slowly by covering the reheated material with powdered charcoal, ashes, sand, or quicklime. Annealing reduces the tensile strength but increases the ductility. Both tempering and hardening increase the elastic limit and the tensile strength but diminish the ductility; however, structural steel is not hardened or tempered, so that these effects will not be further considered.

A few other properties of steel will be briefly referred to: the working of steel at a blue heat from 400° to 650° F. causes brittleness, and may affect the strength of steel field rivets, if these are used in preference to soft iron rivets closed by hand.

Coefficient of Expansion.—Experiments made at the Watertown Arsenal show that the value of the coefficient depends upon the chemical composition of steel. Some of the average results are given in the following table:—

TABLE XII.

Chemical Composition.						Coefficient of expansion per degree Fah., per unit of length.
C.	Mn.	Si.	S.	P.	Cu.	
·17	1·13	·023	·122	·079	·040	·0000067886
·20	·69	·037	·130	·078	·260	·0000068567
·21	1·26	·080	·140	·059	·000	·0000067623
·26	1·07	·110	·096	·080	·047	·0000067476
·28	1·23	·090	·168	·090	·178	·0000067794
·43	·97	·050	·080	·096	·024	·0000066124
·53	·75	·100	·078	·087	·174	·0000064181
·55	1·02	·050	·078	·120	·150	·0000066122
·72	·76	·200	·056	·086	·186	·0000063080

The effect of slow and rapid loading on the ultimate strength of iron and steel at temperatures between 0° and 1000° F. was investigated by a French Commission, vol. ii., pl. xx.; the results are represented in Fig. 31.

The elastic properties of steel at temperatures up to 800° C. for stresses greatly below those necessary for rupture have been investigated by Prof. B. Hopkinson and Mr. F. Rogers, which shows that there is a remarkable change as the temperature increases, in the "time-effect" or "Elastiche nach wirkung" or "creeping". Steel at high temperatures behaves more like indiarubber or glass, in so far as it does not at once recover from the effects of stresses. The decrease in strength with increase of temperature has been proved by many experiments, and it appears to be at the rate of 4 per cent. for every 100° F., attaining a maximum at 400° F., and a minimum at 200° F. Prof. Martens found that the limit of proportionality was lowered as the temperature increased, and could not be determined at 570° F.; the strength diminished from − 4° F. to − 150° F., it increased rapidly up to 480° F., and afterwards diminished rapidly to 6 tons per sq. in. at 1100° F. Forging, hammering, cold rolling, and wire drawing all affect the mechanical properties of steel. In large forgings the material near the surface is denser than in the interior, more especially if made by means of a hammer, and much less so if made by hydraulic pressure. If plates are subjected to cold rolling the elastic limit and tensile strength are increased, but the elongation is diminished. Plates or bars which have been subjected to forging or cold rolling are restored to their normal condition by annealing. When iron or steel is drawn into wire the effect is similar to that of cold rolling. The steel wire used in the cables of the East River Bridge (1902) of New York City has a diameter of 0·192 in., and was specified to have a tensile strength of 200,000 pounds per sq. in., with an elongation of 2·5 per cent. measured on a gauge length of 5 ft., and 5 per cent. on 8 in. The steel from which the wire was drawn was specified to be acid open hearth steel, and not to exceed 0·04 per cent. of phosphorus, 0·03 per cent. of sulphur, 0·5 per cent. of manganese, 0·1 per cent. of silicon, and 0·02 per cent. of copper.

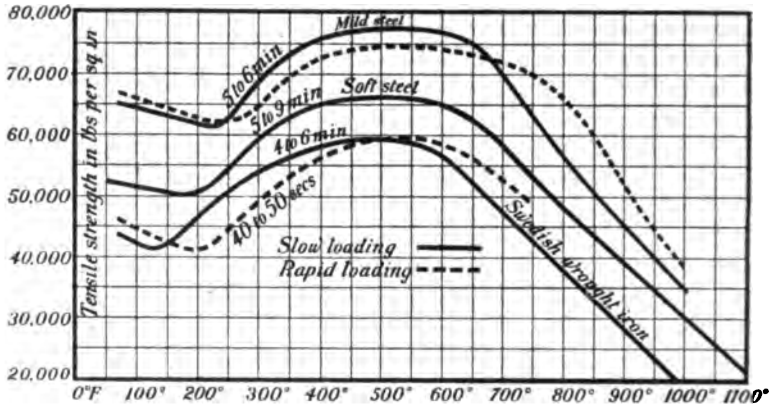

Fig. 31.

Thurston[1] proved that the tensile strength increased and the ductility diminished as the diameter of the wire diminished. The following table gives the results of some of the author's tests of steel wire ropes:—

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[1] *Trans. Am. Soc. C. E.*, vol. iii., p. 212.

TABLE XIII.

RESULTS OF TESTING WIRE ROPE.

Description.	Total number of wires in rope.	Total area of wires in rope.	Tensile strength of wire per square inch. Mean of 7 tests.	Total load required to break the whole rope in pounds.	Tensile strength of rope per square inch of wire.	Ratio of strength of rope to strength of wire.	Number of revolutions obtained in twisting single wire. Mean of 7 tests.	Number of alternate bendings obtained with single wires round radius of ·23 in. Mean of 7 tests.	Result of galvanising test.	Remarks.
Flexible wire rope	36	0·342	179,260	55,150	161,250	90·0	26·6	8·6	Satisfactory	
Do. do.	36	0·342	179,260	59,000	172,510	96·2	26·6	8·6	Do.	
Do. do.	144	0·180	207,000	36,250	201,400	97·1	40·0	38·5	Do.	
Do. do.	144	0·180	215,200	36,600	203,300	94·5	51·5	61·7	Do.	
Do. do.	42	0·326	184,300	44,200	136,600	74·2	29·7	8·4	Do.	Rope been in use for some time.
Do. do.	222	0·627	217,280	113,120	180,320	82·7	37·4	16·5	Do.	
One strand taken from old worn rope . . . . ,	36	0·049	10,600	44,400	89,960	84·8	4·4	6·8	Do.	
Flexible steel wire rope	144	0·181	242,905	30,750	169,926	69·9	29·5	9·6 rad. of ·315″	Do.	Contained hemp cord at centre of rope and each strand. Rope broke near clip.

In all cases the soft iron core wire in each strand was not tested, the mean tensile strength of the remaining steel wires only is recorded. Steel wires have been largely used in suspension bridges, but flexibility, such as is necessary in wire ropes transmitting power, is here a disadvantage. For large suspension bridges the straight wire cable is preferred in America, and it should consist of a number of separate parallel wires gathered and clamped together into a compact cable. The wire should give the following results in testing:—

Minimum tensile strength . .	200,000 pounds per sq. in.
Minimum elongation in 8 in. .	5 per cent.
Minimum elongation in 60 in. .	2½ per cent.

The wire must be capable of being coiled round a rod of its own diameter without cracking. Straight wire cables have been used on the Brooklyn, Williamsburg, and Manhattan Suspension Bridges, New York. The Nuremburg Company in Germany have patented a suspension bridge cable which admits of laying each rope or replacing a rope in a simple manner. Each rope when laid is separate, their connection being effected later on. The patent locked cables made by Messrs. Felten & Guilleaume, Mulheim-on-Rhine, have been tested by the Gustavsburg Bridge Works with satisfactory results. The subject of steel wire suspension bridge cables has been recently treated by Mr. F. Bohny[1] in a very able paper on the theory and construction of suspension bridges.

Effect of Punching and Shearing.—The following experiments recorded in Table XIV. were made by A. F. Hill[2] on open hearth steel, and show the effect of punching and shearing plates of steel as compared with cutting it in a planing machine. It will be observed that both shearing and punching are most injurious to all grades of steel, more especially cold punching. The elastic limit is affected in every case far more than the ultimate strength. The beneficial effect of annealing and tempering on steel plates injured by shearing or punching is also shown in Table XIV.

The Microscopic Structure of Steel.—Steel is not a simple homogeneous union of iron with carbon, manganese, silicon, etc., in various proportions, but it is built up of individual crystals of several varieties differing from each other in chemical composition, size, shape, and mode of occurrence, and moreover the formation of the elements into these phases follows fixed and regular laws.

There is a kind of analogy between the structure of granite and

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[1] F. Bohny, *Theorie und Konstruktion Versteifter Haugebrucken.* Leipsig: Verlag von Wilhelm Englemann, 1905.

[2] *Trans. Engineers Society of Western Pennsylvania for* 1880.

TABLE XIV.

Carbon per cent.	Treatment of specimen.	Tensile strength in pounds, per square inch.	
		At elastic limit.	At rupture.
·03	Cut in planer	49,431	94,396
·03	Sheared	32,370	74,980
·03	Punched	0	63,410
·03	Punched and hammered cold .	0	87,540
·03	Punched, hammered and annealed	55,780	100,410
·04	Cut in planer	63,475	87,695
·04	Sheared	46,900	75,330
·04	Punched	0	68,890
·04	Sheared and annealed . . .	59,350	86,160
·04	Punched and tempered . . .	52,780	103,560
·05	Cut in planer	65,186	84,092
·05	Sheared	51,666	79,000
·05	Punched	0	78,400
·05	Sheared and tempered . . .	60,375	87.293
·05	Punched and annealed . . .	57,960	84,900

that of steel, although the composition of the former is more or less visible to the eye, while that of the latter requires high magnification to make it clear. The microscopic examination of steel is made on highly polished sections which are sometimes etched. The names of Osmond in France; Heine, Wedding, and Behrens in Germany; Sauveur and Howe in America; Arnold, Roberts-Austen, Stead, Sorby, and others in Great Britain, are eminently associated with this branch of knowledge which has such an important bearing on iron and steel of all kinds, that it will be briefly referred to in this work. For full and complete knowledge of the structure of steel the student is recommended to study the works of the authorities above mentioned. The microscope in expert hands is capable of revealing the previous heat history of the steel, to detect the effects of incorrect heat treatment, and the influence of size of grain produced by over-heating on the physical properties of the material, also to discover microscopic defects and flaws.

The following varieties of the constituents of steel have been recognised and described:—

Ferrite, the microscopic particles of pure iron free from carbon and other substances dissolved in it. It is magnetic, very soft, and ductile, but relatively weak with a tensile strength of about 20 tons per sq. in. When much slag is present, as in wrought-iron, and is drawn out into fibres by rolling, the mass as a whole is thereby given a certain kind of pseudo-fibrous structure. Ferrite is not formed when the proportion of carbon exceeds 0·9 per cent. Fig. 32 shows a specimen of basic Martens steel (0·5 per cent. carbon) polished and etched, also magnified about 133 diameters. The light-coloured spots consist of ferrite.

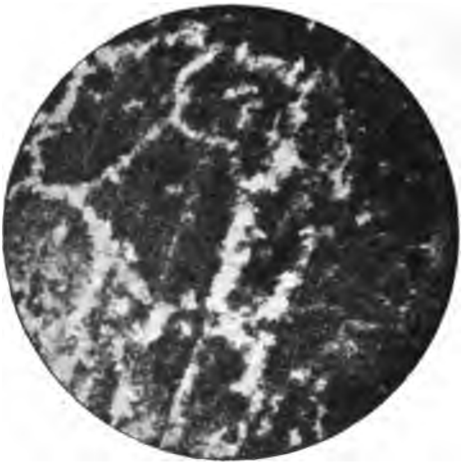

FIG. 32.

Cementite, a definite carbide of iron Fe_3C well defined in high carbon steel, extending throughout the mass. It contains 6·67 per cent. of carbon, is very brittle, harder than hardened steel. The carbon in slowly cooled steel is chiefly or wholly present as cementite in the proportion of 15 per cent. for each per cent. of carbon present. In slowly cooled cast-iron it is probable that nearly all of the combined carbon is present as cementite. If the surface is ground with a soft material the cementite appears in relief. Fig. 33 shows a specimen of cemented steel described by Osmond (1·5 per cent. carbon), polished, etched and magnified to 500 diameters, showing veins of cementite.

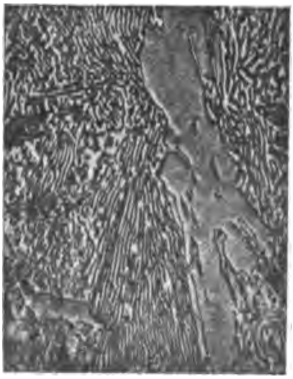

FIG. 33.

Pearlite, according to Howe,[1] is an æolic or quasi-eutectic consisting of interstratified plates of ferrite and cementite, in the ratio of about six parts of ferrite to one of cementite. It requires to be magnified at least 300 diameters, and it occurs in all unhardened steels. If the steel contains about 0·9 per cent. carbon it forms the whole mass; if less carbon is present there is an excess of ferrite; if more carbon is present there is an excess of cementite. Messrs. Seaton and Jude[1] state that "normal mild steel" is a mixture of crystalline ferrite and pearlite grains, more or less of a crudely crystalline form. Fig. 34 shows a characteristic pearlite structure obtained by Osmond from steel containing about 1 per cent. carbon, forged, annealed at 800° C. (1472° F.), polished in relief and magnified about 500 diameters.

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[1] *Iron and Steel and other Alloys* (Howe).

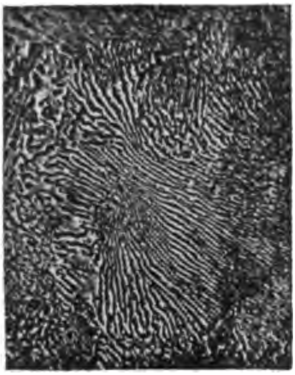

FIG. 34.

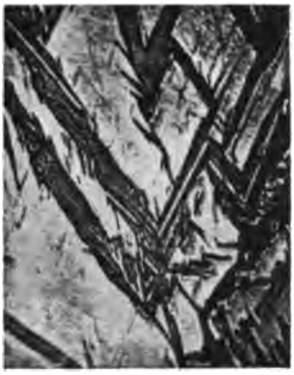

FIG. 35.

Austenite is produced if hard steel (1·5 per cent. carbon) is heated to 1050° C. (1922° F.), and then rapidly chilled in ice-water. It forms a hard brittle mass with a needle-like structure, and is a solid solution of carbon in iron. Fig. 35 shows a specimen described by Osmond, polished and etched at the same time and magnified about 500 diameters. The clear portions show austenite.

Martensite, troosite, and sorbite are transition forms between austenite on the one hand, and ferrite and cementite on the other, and need not be further considered in a brief description.

Graphite, practically pure carbon, is a characteristic component of grey cast-iron.

Slag, a ferrous silicate, is the characteristic component of wrought-iron.

Most of the steel of commerce is of the pearlite series, consisting of pearlite and, where the carbon is less than 0·90 per cent., with excess of ferrite. All rolled girders, channels, bars, rivets, plates, axles, tyres, etc., consist of ferrite and cementite interstratified as pearlite, with an excess of ferrite.

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[1] *Engineering*, 25th November, 1904.

Fig. 36 shows the tenacity, ductility, and mineralogical hardness of iron with varying percentages of carbon—that is, the physical characteristics of these various mixtures of ferrite and Fe_3C.

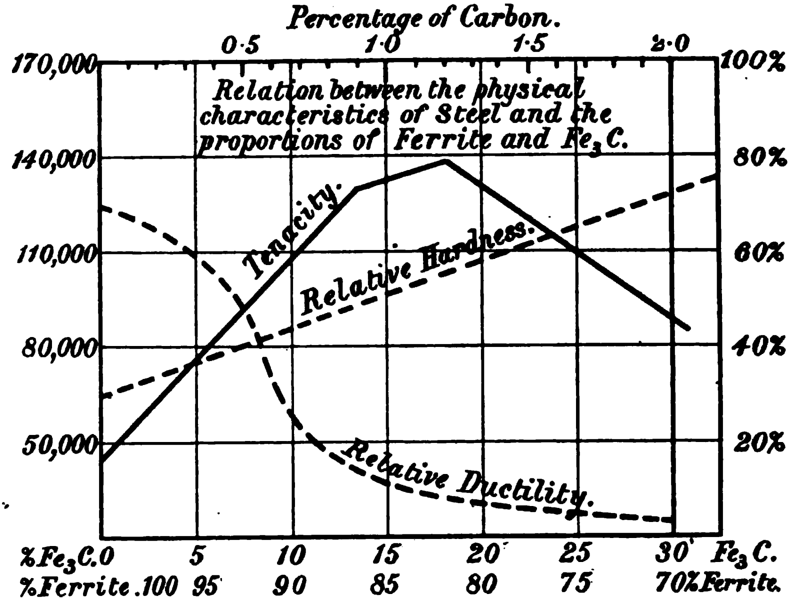

FIG. 36.

This diagram, taken from Howe's work, shows that the softest, weakest, and most ductile of the alloys is pure ferrite, and that there is, with certain exceptions, a decrease in these characteristics with each increase in the proportion of the hard, brittle, and strong Fe_3C. Messrs. Ewing and Rosenheim[1] have shown that the plastic yielding of metals under stress is due to a finite number of slips along cleavage planes of the individual crystals of the metal.

The formation of the constituents in steel containing different percentages of carbon is explained by Osmond by the phenomena which occur during cooling. At certain temperatures called "critical points" recalescence takes place. If the same metal is then heated, the rise of temperature is also irregular near the critical points. The number of critical points and their position vary according to the amount of carbon present, but they are always the same for the same percentage of carbon. Probably chemical or molecular changes take place at the critical points. In the cooling curve of iron and steel there are three critical points,[1] Ar_1, Ar_2, and Ar_3. In the heating curves the points are denoted by Ac_1, Ac_2, and Ac_3, which are a few degrees higher than the critical points in cooling, due probably to a kind of molecular inertia or lag. The critical points are said to define the temperature at which the iron changes allotropically.

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[1] "The Crystalline Structure of Metals," *Phil. Trans. Royal Society*, 1899.

Alpha-iron exists below Ar_2, beta-iron between Ar_2 and Ar_3, and gamma-iron exists above Ar_3. Ar_1 occurs between 650° and 700° C., Ar_2 at 700° C., and Ar_3 occurs between 850° and 900° C.

Gamma-iron or γ-iron is said to be an allotropic non-magnetic condition, very hard. Beta-iron or β-iron is also hard and non-magnetic, while alpha or α-iron is soft and magnetic.

Each critical point is found to be associated with a change in the mechanical properties, the microscopic appearance, the electrical conductivity, the specific gravity and magnetic properties.

Brittleness or toughness of steel of identical chemical composition appears to depend upon its crystalline structure, and hardening, tempering, annealing, and other methods of heat treatment depend upon the alteration in the crystalline structure of the material produced by these processes.

Structure as Revealed by Etched Surfaces.—Prof. Heyn[2] has shown that the structure of iron and steel may be sufficiently revealed to the unaided eye, by means of etching highly polished surfaces, to enable an expert to draw important practical conclusions as to the mechanical properties. The degree of homogeneity among the different components, the distribution of carbon, the presence of local segregations of phosphorus or sulphur may be clearly shown by this process. As it is necessary to employ the microscope to gain a more thorough understanding of the appearances presented, the etching fluid must be of a suitable character for the two methods of examination. Prof. Heyn has found that a solution of cuprammonium chloride in water, 1 : 12, possesses many advantages over the acids. The following tests (Table XV.) are of a brittle boiler plate which fractured along a line of rivets under the cold pressure test; and the illustrations, Figs. 37 and 38, show the relationship between the phenomena of an etched surface and the mechanical properties of the metal.

The bending number in Table XIV. is obtained by means of bars 60 mm. long × 6 × 4 mm. cross-section taken from different portions of the plate to show the properties of the various zonal formations,

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[1] A stands for *Arrêt*, or halt; *r*, for *refroidissement*, or recalescence during cooling; *c*, for *chauffant*, or heating.

[2] *Report on Methods of Etching Malleable Iron for the Visual Investigation of Structure, and on the Lessons to be learnt therefrom*, by Prof. E. Heyn, Royal Testing Laboratory, Gross Lichterfeld W. Berlin: Int. Assoc. for Testing Materials, 1906.

TABLE XV.

Position of specimen.	P per cent.	S per cent.	Yield-point, kilos. per sq. mm.	Ultimate strength, kilos. per sq. mm.	Extension on 200 mm. per cent.	Bending number.
Outer zone . .	0·088	0·04	—	—	—	0 – ½
Along the dark stripes in the inner portion	0·203	0·16	—	—	—	0
Inner portion . .	—	—	—	—	—	0 – ½
Average over the whole section	0·168	0·10	long. 26·3	42·9	23·8	—
			trav. 24·6	42·3	25·4	—

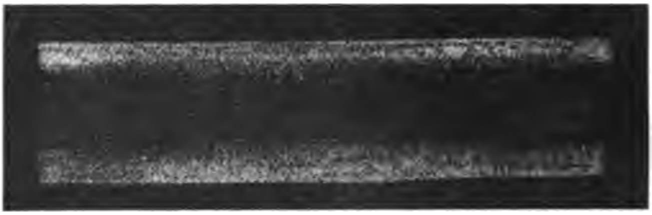

FIG. 37.

such as along the dark stripes in the inner portion of Fig. 37, due to segregations of phosphorus. The bar is notched in the centre with an equilateral notch 0·5 mm. deep, fixed in a vice, and bent through a right angle by blows from a hand hammer. This is termed the first bending, after which the bar is straightened and the process repeated. The bending number, in low carbon steel, varies from 0 to 4, the latter being obtained from the best material. Here also the ordinary tension test failed to indicate brittleness, but the impact test shows how very brittle the plate is, especially along the zone containing segregations of phosphorus.

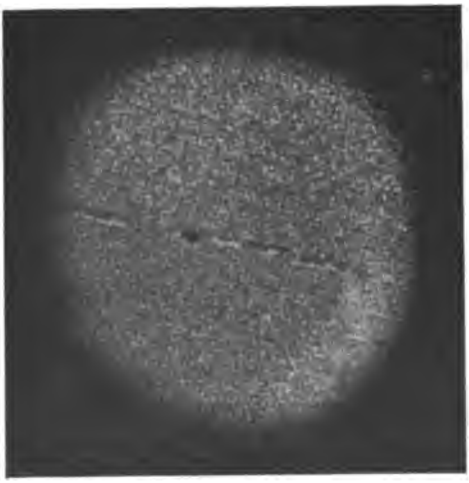

FIG. 38.

The cause of certain phenomena in bending and breaking tests is also shown by etching; thus in Fig. 39 the segregations of phosphorus in a bar of steel fractured by tension, and cut longitudinally, explain the broken surface of the test-piece due to hard veins extending to the surface shown in Figs. 39 and 40.

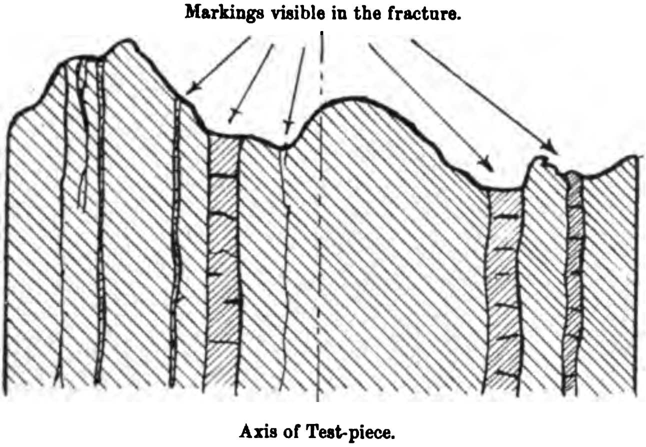

Fig. 39.

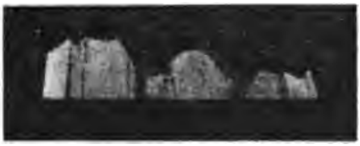

Fig. 40.

The etching process may be applied to rolled steel girders, rails, tyres, axles, and ingots, and similar instructive results obtained.

The Determination of the Safe Working Stresses in Bridges.—It is well known that a moving train passing over a bridge produces a greater effect than the same train standing on the bridge. The increase in the effect produced by live load stresses depends upon their magnitude, speed of application, and their ratio to the mass of the structure affected, being larger for short spans such as floor beams than for main girders or trusses. The writer's experience in testing a large number of bridges[1] shows that the deflections in lattice girders with trains moving at 40 miles an hour, may be 15 per cent. greater than with the same load at rest, for spans of 156 to 159 ft. Plate web girders of 60 ft. span gave an increase in deflection due to speeds of 25 miles an hour of 20 per cent. Prof. Turneaure[2] found that speeds of less than 25 miles an hour produced only a small increase, but at 50 miles an hour the increase may be 50 per cent. for spans of 50 ft., and 25 per cent. for spans of 75 ft. He considers that owing to cumulative effect on spans from 75 to 150 ft., the maximum increase will be from 20 to 25 per cent.

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[1] Royal Commission Railway Bridges, N.S.W., 1885-86.

[2] "Moving Train Load Experiments," *Trans. Am. Soc. C. E.*, June, 1899.

M. Cuenot[1] found that a riveted bridge of 114 ft. span was stressed about 22 per cent. more from speeds of about 16 miles an hour than for a static load. M. Rabut[2] has made an important series of tests of bridges on the western railways of France, and considers that the excess of dynamic over static deflection and stresses is very small for girders of long span, but is considerable for girders of short span, such as stringers and cross-girders. Prof. Howe obtained an increase in deflection of from 7½ to 10 per cent. in spans of 122 and 164 ft., with speeds under 25 miles an hour. Mr. Stone[3] obtained from bridges in India an increase of 50 per cent. for short spans, and 14 per cent. for spans in which the ratio of live to dead load was 4. Prof. Turneaure also found that secondary stresses were high in small girders with shelf angles and in some parts of trusses, and the discrepancy between observed and computed stresses may be greater from this cause than from the dynamic effect of moving loads. It is to be noted that in railway bridges the increased stresses imposed by moving train loads travelling at high speeds are due principally to the action of the unbalanced moving parts of the locomotive, and to the extra load imposed by the vibration of the locomotive on its springs, and also in a lesser degree to the lurching action of the train caused by the rails not being exactly level, the shocks due to inequalities at joints, the effect of flat wheels, and the thrust due to obliquity of the connecting rods of the locomotive. These causes all tend to set up vibrations in the bridge, and for high speeds in main girders of 100 ft. span and over, these vibrations very often become cumulative, owing to the even spacing of the wheels of the train, and increase the deflection of the structure, especially if the impulses of the loads are timed to agree with the natural rate of vibration of the structure. For short spans, such as longitudinal girders of a railway bridge, the vibration is non-cumulative, and the increase in stresses is due to shock or impact solely.

In road bridges the causes producing impact or vibration are less marked, but inequalities in the surface of the roadway must increase the effect of a rolling load to some extent, while considerable vibration may be set up by trotting horses or a mob of cattle.

Three methods are in use to-day for the determination of the working stresses in structures:—

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[1] *Annales des Ponts et Chaussées*, July, 1891.

[2] *Engineering*, 13th Aug., 1897; *Le Génie Civil*, vol. xxii., 1892, p. 88.

[3] *Proc. Am. Soc. C. E.*, May, 1898, p. 364.

(*a*) A definite working stress is decided upon, which, in the judgment of the designer, provides for the effect of the live load and other causes tending to increase the stresses.

(*b*) A working stress is determined from formulæ based upon repetitive tests, using constants depending on the material.

$b = c\left(1 + \frac{1}{2}\,\frac{\text{min.}}{\text{max.}}\right)$; here 50 per cent. greater stress is allowed for the dead than the live load.

$b = c\left(1 + \frac{\text{min.}}{\text{max.}}\right)$; this allows 100 per cent. increase for dead over live load stresses.

$b = \frac{f}{6}\left(2 - \text{R}^2\right)$; proposed by E. H. Stone, where f denotes the ultimate statical strength of the material, and R the ratio of the range of stress to the total stress.

(*c*) A definite allowance is made for the effect of the impact of the live load, and the equivalent dead load determined. It is, therefore, only necessary to determine in this case the safe dead load stress in tension, in compression, in shear and in bearing.

In regard to (*a*), a definite working stress for all parts of the structure in tension has been fixed by Mr. Theodore Cooper[1] for medium steel, as follows:—

Floor-beams, hangers, and similar members liable to sudden loading . . .	6,000 pounds per sq. in.
Solid rolled beams used as cross-girders or stringers	10,000 ,, ,,
Bottom-flanges of riveted cross-girders, longitudinal track stringers, and net-section .	10,000 ,, ,,
Longitudinal lateral sway bracing for wind loads	18,000 ,, ,,
Longitudinal lateral sway bracing for live loads	12,000 ,, ,,

For bottom chords, main diagonals, counters and long verticals in trusses and other structures, Mr. Cooper allows 20,000 pounds per sq. in. for the dead load stresses in railway bridges, and 22,000 pounds per sq. in. in highway bridges, while for live loads he allows one-half of these values in every case. Again, for the compression members special formulæ are given for the various portions of the bridge, and in every case the allowance for live load stresses is only one-half that allowed for dead load stresses. If soft steel is used a deduction of 10 per cent. is made for tension stresses, and 15 per cent. for compression stresses on the values given for medium steel.

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[1] *General Specification for Steel Railroad Bridges*, 1901.

In regard to (*b*) the equation proposed by the French Government for steel bridges is, for tension members—

$$b = 5{\cdot}08\left(1 + \tfrac{1}{2}\,\frac{\text{min.}}{\text{max.}}\right).$$

If the ratio of the dead to the total load stress in a tension member of a bridge is as one to ten, then—

$$b = 5{\cdot}08\ (1 + \tfrac{1}{20}) = 5{\cdot}334 \text{ tons per sq. in.}$$

The formula proposed by Mr. E. Herbert Stone,[1] based on a large number of experiments on Indian bridges, for steel bridges is—

$$b = 9 - 5\mathrm{R}^2 \text{ where } \mathrm{R} = \frac{\text{range of stress}}{\text{total stress}}.$$

If the ratio of the dead to the total load stress is as one to ten, then—

$$\mathrm{R} = 0{\cdot}9,\ \mathrm{R}^2 = 0{\cdot}81\ \therefore b = 9 - 5 \times 0{\cdot}81 = 4{\cdot}95 \text{ tons per sq. in.}$$

In regard to (*c*) two kinds of impact formulæ have been proposed—

$\mathrm{I} = \dfrac{40{,}000}{l + 500}$ (Waddell).[2] I = percentage allowance for impact.

$\mathrm{I} = \mathrm{S}\left(\dfrac{300}{l + 300}\right)$ (Schneider).[3] I = increment for impact. S = live load stress.

$\mathrm{I} = \mathrm{S}\left(\dfrac{300}{l + 500}\right)$ (Hobson in Victoria Falls Bridge).

These are similar, and in each case the percentage of impact to be added to the live load depends upon the length of the span; thus for 500 ft. spans, Waddell's formula gives 40 per cent. increase, and Schneider's 37½ per cent., while Hobson gives 30 per cent. increase.

A formula adopted by J. W. Schaub, 1900, proposed by H. S. Pritchard, 1895, is—

$$\mathrm{I} = cl\left(\frac{l}{l + d}\right),$$

where I = the increment for the impact effect of the live load,
l = live load,
d = dead load,
c = a constant usually taken as ·75 for train loads, ·30 for rolling loads on road bridges, and ·15 for a crowd of people.

This formula takes into account the inertia of the structure as resisting the dynamic effect of the live load, and differs from the Waddell, Schneider, Hobson formulæ, which are based on the length of span or member under load, and not upon its mass. The allowance of ·75 for train loads is based on observed deflections on bridges and cannot be very far wrong. The method of using this formula will now be illustrated.

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[1] "The Working Stresses in Railway Bridges," *Trans. Am. Soc. C. E.*, vol. xli., 1899.

[2] *De Pontibus*, 1899.

[3] American Bridge Company General Specifications for Railroad Bridges.

Let the working stress for dead load be taken at 17,000 pounds per sq. in. for medium steel, which gives a factor of safety of 3½ on the lower limit.

$$\text{Let } d = 0, \text{ then } \frac{l}{l + d} = 1{\cdot}0.$$

In this case, the impact allowance is—

$$\mathrm{I} = {\cdot}75l.$$

The static working stress in tension is therefore—

$$b = \frac{17000}{1{\cdot}75} = 9714 \text{ pounds per sq. in.}$$

If the live and dead load are equal—

$$\mathrm{I} = {\cdot}75l \times \tfrac{1}{2} = {\cdot}375l.$$

The static working stress in tension is—

$$b = \frac{17000}{d + 1{\cdot}375l} = \frac{17000}{{\cdot}5 + 1{\cdot}375 \times {\cdot}5} = 14{,}316 \text{ pounds per sq. in.}$$

Take the case of a longitudinal railway stringer in a bridge having panel lengths of 30 ft. In this case d = 260 pounds per ft., l = 3600 pounds per ft. :—

$$\frac{l}{l + d} = 0{\cdot}93,$$

$$\mathrm{I} = {\cdot}75 \times {\cdot}93l = {\cdot}7l,$$

$$b = \frac{17000}{0{\cdot}07 + 0{\cdot}93 \times 1{\cdot}7} = 10{,}303 \text{ pounds per sq. in.}$$

This result corresponds fairly well with Mr. Cooper's allowance of 10,000 pounds per sq. in. for the tension members of railway stringers. The French formula gives 5·334 tons or 11,948 pounds per sq. in. The diagram,[1] Fig. 41, was constructed to compare the working stresses proposed in a bridge over Sydney Harbour with those given by other authorities for similar conditions. This diagram is very instructive, and shows clearly the practice of the best-known authorities in bridge engineering at the present time.

CONCLUSIONS ON WORKING STRESSES.

(*a*) In regard to Cooper's specifications, the working stresses allowed may certainly be safely followed, as they are based upon the wide knowledge and experience of an eminent bridge engineer; moreover,

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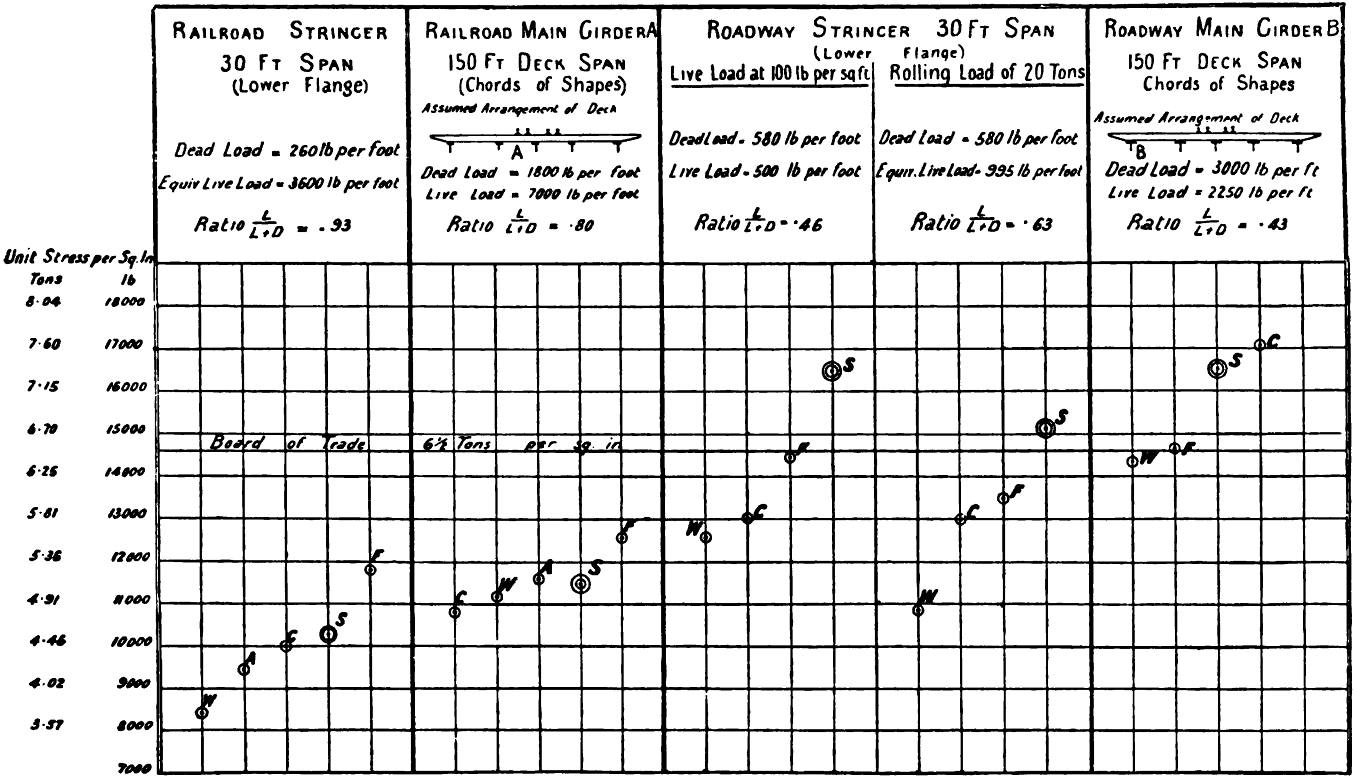
[1] *The Determination of Working Stresses in Bridges*, by H. H. Dare, M.E., and the author. *Proc. Sydney University Engineering Society*, October, 1901.

COMPARISON OF UNIT STRESSES IN TENSION FOR VARIOUS SPECIFICATIONS.

(*These are the Statical working stresses per sq. in. after allowing for Impact.*)



A=American Bridge Co. Specification, 1900 (Schneider). **C**=Theodore Cooper's Specifications, 1896 and 1901. **F**=French Government Specification, 1892. **W**=J. A. L. Waddell's Specification, 1899. **S**=Sydney Harbour Bridge Specification.

NOTE.—The Dead Load weights given above have been calculated approximately from the design of deck embodied in specification for Sydney Harbour Bridge.

FIG. 41.

they represent the best American practice. It does not appear correct, however, to assume that the live load is always twice as effective as a dead load in bridges, since the suddenly applied load to a beam produces twice the deflection as the same load gradually applied for an instant only. In railway bridges the deflections of the bridge as a whole, or the extensions of any particular member, so far as these have been ascertained, are never twice as great under live load stress, as the bridge appears to recover from the first effect of the live load as the train advances and covers the bridge. The dynamic effect is, however, most probably greater than the deflections, as these are proportional merely to the average vertical movements, some of which will be greater and others less than these, while the horizontal movements are not included at all.

(*b*) The formulæ based upon repetitions of stress experiments at short intervals can hardly be taken to represent the effect of moving train loads which occur at much longer intervals. Moreover, all recent experiments prove that the effect of repetitions of stress is much greater the greater the rapidity of the applications. It is also known that the metal stressed beyond its elastic limit recovers, to some extent, its original molecular condition when the interval of time between the applications is sufficiently long. It would therefore appear that the effect of repetitions of stress from moving train loads is much less than that produced in the Wöhler-Bauschinger and similar experiments. The dynamic effect of moving train loads on bridges is, however, much greater than could possibly occur in the Wöhler machines where any dynamic effect must necessarily be very small. The effect of repeated stresses in causing rupture, with stresses less than the statical strength of the material, cannot be applied to bridges in which the working stress is less than the vibrating strength of the material, *i.e.*, less than one-third of the statical strength, as stated by Martens: "Rupture occurs only when the stress repeated exceeds a certain amount. . . ." It therefore appears that the effect of repeated stresses in bridges, where these are well within the true elastic limit of the material, is much less important than the dynamic effect of these stresses.

(*c*) Impact formulæ are in much more general favour with designing engineers as they possess one great advantage over formulæ based on range of stress—thus: impact formulæ provide for a definite increase in the live load stresses reducing them to equivalent dead load stresses. They are then treated as dead load, and divided by the allowed working stress. This stress is constant throughout the bridge, for the same kind of stress, whether tension, compression, shear or bearing pressure on pins or rivets. In formulæ based on range of stress, the ratio of the minimum to the maximum is not always that of the dead to the live load; the absolute minimum stress sometimes being due to the dead load stress, less a stress of opposite sign due to the partially distributed live load. Hence the ratio of minimum to maximum varies throughout the members of the bridge, and a different working stress must be determined for each case, involving additional labour.

No account is taken in any of these methods for the effect of initial stresses in the material caused by cold straightening during manufacture, or increase in stresses due to oxidation, or unequal temperature, any of which may cause the metal to exceed its elastic limit at some point. The employment of a material which stretches regularly beyond its elastic limit, ensures the equal distribution of stresses over the member over-stressed, by the slight yielding which occurs locally.

While it is possible to determine the working stresses by either of the three methods discussed, using suitable constants for the material, the more rational method appears to consist of an impact allowance reducing the live load to an equivalent dead load, and the best formula of this kind is probably that proposed by Pritchard. It does not follow, however, that the values of the coefficient c should be taken as 0·75 for train loads, and it may be increased for the shorter spans with advantage. In the case of cross-girders of bridges and longitudinal stringers, which receive the impact of the live load directly, the value of the coefficient may be taken as equal to unity. Even in this case the impact allowance is less than that allowed by Mr. Cooper.

CHAPTER II.

TIMBER.

TIMBER has been extensively employed as a material in engineering construction, and there is good reason to believe that it will continue to be a most important factor in the engineering work of the future. It is necessary for the engineer to know the strength of the various kinds of timber in common use, its durability and the various characteristics which affect its use. It has long been known that the quality is influenced to a considerable extent by the conditions of its growth, the climate, kind of soil, the age of the tree, the time of felling, the extent of the seasoning as shown by the percentage of moisture, etc. Investigations in connection with the physical properties of timber date back to the commencement of the eighteenth century, and from time to time results have been published. Among the earlier investigators may be mentioned Messrs. Chevaudier and Wertheim (1848), Dr. Nördlinger (1860), Peter Barlow (1883), and more recently Professors Bauschinger, Tetmajer, Martens, Rudeloff, B. E. Fernow, Johnson, Lanza, Bovey, and Hatt.

The most valuable results for engineering practice are those obtained from testing full-size pieces such as are used in construction, in which knots, cracks, crooked grain, and other defects occur to such an extent as may be expected in the timber obtained in the ordinary market. Small specimens in which defects and irregularities have been carefully avoided give results much higher than those obtained from testing full-size pieces of average quality, and are misleading when applied to timber under practical conditions.

Rules for testing timber have been drawn up by Prof. Rudeloff, and adopted by the International Association for Testing Materials. Prof. W. Kendrick Hatt has also proposed rules for testing timber in connection with the Forest Service, U.S. Department of Agriculture. These reports are worthy of careful study, but the object of this chapter is to give the results of tests of timber rather than the precautions necessary to be observed in obtaining accurate results in testing.

Tension Tests.—In testing timber it is difficult to get trustworthy results with soft woods, even when all known precautions are rigidly applied. In Prof. Bovey's tests the increments of extension up to the point of fracture were almost directly proportional to the increments of load. The total extension is greatest during the first loading, and the extensions due to successive loadings continually diminish up to a minimum limit; successive unloadings produce a set which continually increases, but at a diminishing rate to a maximum limit. The presence of knots produces great irregularities in the readings, and the same may be said of curly or oblique grain caused by a knot. The writer's investigations on the Australian timbers showed that uniform extensions only occurred when the fibre was straight and parallel to the axis of the test-piece. Tension tests are interesting and they have a scientific value, if carefully conducted, but the results are practically valueless to the engineer for the following reasons. They are necessarily made on small pieces rarely exceeding 1 sq. in. in cross-sectional area. Whenever timber is used as a tension member in a structure, as in a tie bar of a roof, or in trusses with joints made by means of bolts and plates, failure will certainly not occur by pure tension, but it may occur by the shearing of the timber, the pulling out of the bolts or fastenings, or by bending. The resistances developed are all less than the tensile strength of the timber, and it is, therefore, the strength of the joints and connections which are of importance, as these must necessarily fail in almost every case long before the tensile strength of the timber is fully developed. The average strength of spruce and red pine timber cut from the circumference and heart of the tree, is about 10,000 pounds per sq. in., and most Australian timbers exceed 20,000 pounds per sq. in.

Compression Tests.—The value of the compressive strength of timber has been shown by Bauschinger and Rudeloff to be a measure of the quality of the timber, and it is used to ascertain the effect of the conditions of the growth of the tree on the strength of the timber. The engineer requires to know the compressive strength and elasticity in designing timber compression members. Lanza's tests of American timbers 12 ft. and 2 ft. long, of circular and rectangular sections, gave the following results :—

TABLE XVI.

Name of timber.	Sectional area in sq. in.	Ultimate strength in pounds per sq. in.	Coefficient of elasticity in pounds per sq. in.
Yellow pine	42 to 102	4544	1,996,351
White oak	32 to 93	3470	1,398,908
Old and seasoned white oak .	23 to 87	3957	1,817,539

The columns all failed by direct crushing, and not by lateral flexure as in long columns. Mr. Ely's tests at the Watertown Arsenal show the effect of the length to the least side of the rectangular cross-section, denoted by l/d. f denotes a constant the value of which is given in the following table in pounds per sq. in. a is the area of the cross-section, then—

$$\text{Breaking strength} = fa.$$

TABLE XVII.

White pine.		Yellow pine.	
$\frac{l}{d}$	f	$\frac{l}{d}$	f
0 to 10	2500	0 to 15	4000
10 to 35	2000	15 to 30	3500
35 to 45	1500	30 to 40	3000
45 to 60	1000	40 to 45	2500
		45 to 50	2000
		50 to 60	1500

The following tables give the results of tests made by Prof. Hatt of the Bureau of Forestry, U.S. America, on red fir grown on the Pacific coast (*Pseudotsuga taxifolia*), otherwise called Oregon pine or Douglas spruce, the western hemlock (*Tsuga hetrophylla*), the long leaf pine (*Pinus palustris*), and the loblolly pine (*Pinus tæda*).

Table XVIII. contains the results of tests in compression parallel to the fibre, and Table XIX. contains the results of tests made at right angles to the fibre. The loads in the latter case were applied through a steel plate 4 in. long, bearing on the test piece supported on the compression plate of the machine, and the amount of yielding under the 4-inch plate is determined at the elastic limit. The value found in this manner represents the resistance of timber against the bearing pressure of a washer, or of a rail upon a sleeper.

TABLE XVIII.

COMPRESSIVE STRENGTH PARALLEL TO THE FIBRE (HATT).

Species.	Where grown.	Dimensions.	Grade.	Number of tests.	Stress at elastic limit.	Stress at rupture.	Moisture.	Number of rings per in.	Weight.
					Lbs. per sq. inch.	Lbs. per sq. inch.	Per cent.		Lbs. per cubic ft.
Red fir	A	5″ × 5″	All—						
			Average	80	3590	4788	20·7	15·5	29·0
			Maximum		5280	7070	22·7	30·0	36·2
			Minimum		1970	2780	17·5	4·0	22·9
Red fir	B	5″ × 5″	All—						
			Average	74	2590	3376	26·3	—	26·0
			Maximum		3760	4515	36·5	—	31·6
			Minimum		1270	1620	20·0	—	22·1
Red fir	C	5″ × 5″	All—						
			Average	63	3840	4886	20·8	19·4	28·4
			Maximum		5220	6100	28·3	36·0	33·2
			Minimum		2500	3440	15·7	7·5	23·8
Western hemlock	Oreg.	5″ × 5″	All—						
			Average	57	2560	3353	32·0	9·3	25·7
			Maximum		3930	4620	47·1	16·0	30·6
			Minimum		1520	2660	22·2	5·0	22·5
Loblolly pine	Va.	8″ × 8″	Square edge—						
			Average	13	1346	1988	74·0	—	26·4
			Maximum		1950	2485	104·6	—	31·9
			Minimum		819	1362	61·9	—	20·0
Loblolly pine	S. C.	8″ × 8″	Square edge—						
			Average	24	2748	3862	39·0	—	29·6
			Maximum		4490	5600	84·0	—	40·0
			Minimum		1925	3035	22·6	—	24·4
Longleaf pine	Ga.	4″ × 6″	Merchantable—						
			Average	44	3650	4862	26·9	—	35·7
			Maximum		5200	5950	34·8	—	46·8
			Minimum		2391	3290	21·7	—	31·0

TABLE XIX.

COMPRESSIVE STRENGTH AT RIGHT ANGLES TO THE FIBRE (HATT).

Species.	Where grown.	Dimensions.	Grade.	Number of tests.	Fibre stress at elastic limit.	Moisture.	Dry weight.
					Lbs. per sq. inch.	Per cent.	Lbs. per cubic ft.
Red fir .	A, B, C .	4″ × 4″ × 16″	All—				
			Average .	142	597	[1]19·5	[1]28·8
			Maximum .		1300	22·4	35·9
			Minimum .		402	17·5	23·0
Western hemlock .	Oregon .	4″ × 4″ × 16″	All—				
			Average .	57	478	34·1	26·7
			Maximum .		988	53·3	34·1
			Minimum .		234	21·0	21·9
Loblolly pine .	Virginia .	8″ × 8″ × 28″	Square edge—				
			Average .	13	412	74·7	26·1
			Maximum .		866	104·6	31·9
			Minimum .		195	61·9	20·2
Loblolly pine .	S. Carolina	8″ × 14″ × 28″	Square edge—				
			Average .	23	623	44·8	29·7
			Maximum .		937	117·0	40·0
			Minimum .		375	23·5	24·4
Longleaf pine	Georgia .	4″ × 4″ × 16″	Merchantable—				
			Average .	22	616	25·1	36·0
			Maximum .		875	30·9	43·2
			Minimum .		375	21·7	31·0

[1] Results of fifty-one tests.

The following table is given by Prof. Burr[1] representing timber as actually used in America for timber structures; it was taken from a report made by the Committee of the Association of Railway Superintendents of Bridges and Buildings. The values of the working stresses according to Burr are those in common use in the design of structures.

TABLE XX.

TIMBER IN COMPRESSION.

Kind of timber.	Ultimate strength pounds per sq. in.			Average working stresses pounds per sq. in.		
	With grain.			With grain.		
	End bearing.	Columns under 15 diameters.	Across grain.	End bearing.	Columns under 15 diameters.	Across grain.
White oak	7000	4500	2000	1400	900	500
White pine	5500	3500	800	1100	700	200
Southern longleaf or Georgia yellow pine . . .	8000	5000	1400	1600	1000	350
Douglas Oregon and yellow fir	8000	6000	1200	1600	1200	300
Washington fir or pine (red fir)						
Northern or shortleaf yellow pine	6000	4000	1000	1200	800	250
Red pine	6000	4000	800	1200	800	200
Norway pine	6000	4000	800	1200	800	200
Canadian (Ottawa) white pine	—	5000	—	—	1000	—
Canadian (Ontario) red pine .	—	5000	—	—	1000	—
Spruce and eastern fir . .	6000	4000	700	1200	800	200
Hemlock	—	4000	600	—	800	150
Cypress	6000	4000	700	1200	800	200
Cedar	6000	4000	700	1200	800	200
Chestnut	—	5000	900	—	1000	250
California red wood . .	—	4000	800	—	800	200
California spruce . . .	—	4000	—	—	800	—

The following table has been compiled from experiments made by Kirkaldy :—

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[1] *Elasticity and Resistance of Materials.*

TABLE XXI.

Name of timber.	Sectional area in sq. in.	Ratio $\frac{l}{d}$	Compressive strength in pounds per sq. in.	Coefficient of elasticity in pounds per sq. in.
English oak . .	25	9	3,542	1,786,600
„ „ . .	54·5	5 to 22	3,501	—
„ „ . .	83 to 144	8 to 10	3,431	—
Pitch pine . . .	80	11	4,339	1,539,000
„ „ . . .	75	5	4,569	—
„ „ . . .	25	9	4,934	—
White Riga fir . .	169	19	1,960	—
Red Dantzig „ . .	178	18	1,742	—
Dantzig „ . .	83	11	2,588	—
„ „ . .	100	5	2,914	—
„ „ . .	25	9	4,704	2,630,000
Oregon pine .	25	9	6,586	1,923,000
Norway spruce . .	25	9	5,376	2,040,800
Baltic red . . .	75	5	2,445	1,163,000
English ash . .	25	9	3,025	1,613,000
Demerara greenheart.	25	1·8 to 18	10,229	4,301,800
„ „ .	50	1·2 to 18	8,430	3,744,513

For further information on timber columns, see Chapter XII.

Transverse Strength.—Tests for the transverse strength are usually made on beams supported at each end and loaded in the centre; then—

$$W = \frac{2fbd^2}{3l}\,; \text{ and } f = \frac{3Wl}{2bd^2};\ E = \frac{Wl^3}{4vd^3b}\,; \text{ and } v = \frac{Wl^3}{4Ed^3b}.$$

v = the deflection; E = the co-efficient of elasticity; W = the load applied; b = the breadth; d = the depth; and l = the span of the beam. f denotes the stress at the extreme fibres, provided that the deflections are proportional to the loads producing them. If W is the breaking load, then f denotes the modulus of rupture. The intensity of horizontal shearing stress along the neutral axis is—

$$\sigma_s = \frac{3}{2}\frac{S_h}{bd},$$

$$\text{and } \frac{d}{l} = \frac{2S_h}{f}.$$

If S_h and f are known, we can determine the ratio of the depth to the span in a beam that will be on the point of shearing at the same time as the maximum value of f is just reached.

The following table gives a summary of Prof. Bauschinger's tests of the more important coniferous woods obtained from different districts of Bavaria:—

TABLE XXII.

Name of timber.	Average percentage of moisture.	Breadth and depth in inches, approx.	Span in feet.	Modulus of rupture, pounds per sq. in.	Coefficient of elasticity, pounds per sq. in.
Pine (summer felled) .	22	7×7	8·2	6770	1;539,639
,, (winter felled) .	33	7×7	,,	6418	1,468,524
Spruce (summer felled)	28	$10\frac{1}{2} \times 10\frac{1}{2}$ to 7×7	,,	5327	1,412,818
,, (winter felled) .	28	11×11 to 7×7	,,	5470	1,402,502

The following table gives a summary of Prof. Lanza's tests of large size beams of American timber :—

TABLE XXIII.

Name of timber.	No. of tests.	Breadth and depth of beam in inches.	Span in feet.	Modulus of rupture, pounds per sq. in.	Coefficient of elasticity, pounds per sq. in.
Yellow pine beams—					
Maximum . .	51	$9 \times 13\frac{3}{4}$	24·5	11,360	2,286,000
Minimum . .		$3 \times 12\frac{1}{8}$	6·4	3,963	1,162,467
Mean . . .		—	18·0	7,486	1,757,900
White oak beams—					
Maximum . .	36	$6\frac{5}{8} \times 12\frac{1}{4}$	20·0	8,850	1,777,500
Minimum . .		3×9	9·6	3,535	672,724
Mean . . .		—	16·9	5,863	1,131,100
White pine beams—					
Maximum . .	37	$6\frac{1}{16} \times 12\frac{1}{16}$	20·0	7,251	1,565,000
Minimum . .		$2\frac{7}{8} \times 9\frac{3}{4}$	14·0	2,456	727,000
Mean . . .		—	16·6	4,451	1,122,000
Spruce beams—					
Maximum . .	68	6×12	20·0	8,120	1,899,800
Minimum . .		$1\frac{3}{4} \times 6\frac{3}{4}$	6·7	2,828	897,961
Mean . . .		—	14·5	5,046	1,332,451
Kiln-dried white pine beams—					
Maximum . .	8	$2\frac{1}{16} \times 13\frac{3}{8}$	15·0	7,014	1,505,000
Minimum . .		$2 \times 12\frac{1}{16}$	13·0	2,836	752,300
Mean . . .		—	—	5,482	1,183,037
Hemlock beams—					
Maximum . .	17	$4\frac{1}{8} \times 12$	17·0	6,535	1,327,200
Minimum . .		$3 \times 9\frac{7}{8}$	11·0	1,531	412,670
Mean . . .		—	—	3,825	922,250

Prof. Hatt's tests of beams for the Bureau of Forestry were selected in accordance with the rules for the classification and grading of timber recently adopted by various timber companies in America.

The grades tested were termed "select," "merchantable" and "seconds". The "select" consists of sound, strong timber well sawn; the "merchantable" is suitable for good substantial constructional purposes; the "seconds" is not suitable for substantial constructional purposes, but suitable for inferior work. The moisture in the beams was determined by the section method, and the percentage calculated with reference to the weight of the dry wood. Loads were applied continuously at a speed of one-quarter inch deflection per minute for beams 6 x 8 in cross-section, and at a speed of one-eighth inch per minute for beams 8 x 16 in cross-section. The elastic limit recorded is said to be the true limit of proportionality in the deflections.

The table on pp. 68, 69 gives the results of testing all grades. In Prof. Hatt's tables the results are given in detail as obtained in testing "selects," "merchantable" and "seconds" separately, but they are indicated in Table XXIV. by maximum, average and minimum.

The results given for Western hemlock do not include "selects"—only "merchantable" and "seconds" have been summarised. The results given for loblolly pine are for "seconds" only, and the results for longleaf pine are for "merchantable" only.

The moisture content is different in green timber of various species; for instance, about fifty per cent. of the dry weight in the case of red fir, and up to 100 per cent. in the case of loblolly pine, so that the same moisture contents do not indicate an equal degree of seasoning. Again, the red fir seasons more rapidly in the dry climate of California than does the loblolly pine in the moist climate of the Atlantic coast. Prof. Hatt draws attention to the large percentage of moisture contained in well-seasoned timber, and states that a stick of longleaf pine, 10 x 12, after drying for one year in the Washington yard, contained 35 per cent. of moisture; that sticks of loblolly pine from Virginia, 8 x 8 in cross-section, after drying for two years, contained 34 per cent. of moisture. The laws expressing the relation of strength and moisture in the case of small sticks, however, do not necessarily apply to large sticks.

Influence of Time on Tests of Timber.—In testing timber beams loaded to produce a definite deflection and allowing the load to remain on for several weeks, it will be observed that the deflection gradually increases. Lanza subjected beams to a load at the centre producing a fibre stress of about 1100 pounds per sq. in. on a span of 20 ft., and observed the deflection when the load was first applied, and again six months afterwards. The beams were unseasoned as received, but were allowed to season under load in the laboratory in steam heat. The deflections in all cases were found to be about twice as great at the end of six months as when the load was first applied.

TABLE XXIV.

CROSS-BREAKING STRENGTH OF LARGE STICKS (HATT).

Name of timber.	Cross-section, inches.	Span in feet.	All grades.									Remarks.
				Number of tests.	Fibre stress at elastic limit.	Modulus of rupture.	Modulus of elasticity.	Longitudinal shear.	Moisture.	Number of rings per inch.	Dry weight per cubic foot.	
					Pounds per sq. in.	Pounds per sq. in.	1000 pounds per sq. in.	Pounds per sq. in.	Per cent.		Pounds.	
Red fir . . .	6 × 8	16	Average	29	5740	7,775	1835	200	19·7	16·7	28·95	
			Maximum		9210	10,910	2665	517	22·8	33·3	35·85	
			Minimum		2710	4,380	1230	79	17·5	4·6	21·55	
Red fir . . .	8 × 16	16	Average	8	5450	7,830	1824	297	22·6	13·4	30·5	
			Maximum		6750	8,680	2200	345	26·5	18·8	35·4	
			Minimum		4440	5,710	1365	228	19·0	6·2	25·2	
Red fir . . .	6 × 8	16	Average	36	3580	5,075	1234	115	27·7	12·4	24·19	
			Maximum		5150	7,534	1710	161	36·5	26·0	38·7	
			Minimum		1165	1,985	762	56	20·5	5·0	20·6	
Red fir . . .	8 × 16	16	Average	18	3860	5,098	1327	212	24·4	6·4	31·0	
			Maximum		5010	7,385	1620	318	30·5	12·0	36·9	
			Minimum		2550	2,940	872	117	20·8	3·0	26·0	
Red fir . . .	6 × 8	16	Average	24	5560	8,030	1850	160	21·2	17·9	29·1	
			Maximum		7210	11,000	2180	215	23·8	25·9	30·9	
			Minimum		1980	3,770	1510	74	17·6	8·0	26·2	
Red fir . . .	8 × 16	16	Average	6	4720	6,850	1517	333	25·3	26·8	26·1	
			Maximum		5220	7,260	1690	395	29·2	29·4	27·2	
			Minimum		4460	6,430	1365	283	23·0	22·7	25·4	

Western hemlock from Oregon .	6 × 8	7	Average	20	4030	5,742	1238	271	31·0	10·2	26·9	Selects not included.
			Maximum		5810	7,190	1597	348	42·4	19·2	28·4	
			Minimum		2650	4,250	1010	188	23·6	5·9	23·6	
Do. .	8 × 16	16	Average	10	3870	5,115	1308	201	34·6	10·1	26·6	
			Maximum		4450	6,060	1575	246	51·2	14·7	29·0	
			Minimum		2430	3,900	1163	113	23·6	6·1	24·7	

Square edge.

Loblolly pine (1) from Virginia .	8 × 8	6 to 15	Average	15	1847	3,281	729	104	72·6	3·1	26·3	Green.
			Maximum		3185	4,720	1150	246	104·6	3·5	31·9	
			Minimum		878	1,545	356	33	61·9	2·5	20·0	
Do. (2)	8 × 8	6 to 15	Average	13	3089	4,874	1164	140	23·3	5·1	28·8	Air dried.
			Maximum		5300	8,950	1728	348	34·7	8·8	36·2	
			Minimum		674	1,470	606	45	17·8	2·7	23·1	
Do. (3)	4 × 8	10	Average	12	2078	5,100	1192	156	46·3	6·4	29·1	Air dried then soaked.
			Maximum		3760	6,240	1640	190	77·7	10·0	32·5	
			Minimum		1020	3,520	842	117	33·4	4·0	22·5	
Do. (4)	4 × 8	10	Average	8	3384	6,319	1219	196	19·4	7·9	29·4	Air dried one year.
			Maximum		4900	8,370	1695	275	21·6	10·3	33·1	
			Minimum		2400	3,640	894	120	14·5	6·0	23·8	
Do. (5) from South Carolina .	8 × 14	16	Average	20	3610	6,187	1479	258	37·2	6·0	31·2	Air dried.
			Maximum		5210	8,680	1920	322	59·2	8·0	36·9	
			Minimum		2460	4,450	932	146	26·1	4·0	25·0	
Longleaf pine from Georgia . .	10 × 12	16	Average	22	5581	8,384	1820	252	27·3	18·0	42·9	
			Maximum		9601	11,410	2920	330	34·5	29·0	49·4	
			Minimum		3547	4,836	1167	151	20·0	11·0	31·4	
Do. from South Carolina . .	6 × 10 to 10 × 8	16	Average	4	3697	7,212	1611	296	23·3	13·5	37·6	
			Maximum		4620	8,640	1875	366	26·1	18·0	47·5	
			Minimum		3080	6,070	1230	238	18·1	10·0	33·1	

A large number of beams were tested in this way with similar results. It is clear, then, that if we require to know the deflection of a beam after it has been loaded for a considerable time, we must take the coefficient of elasticity at about one-half of the value obtained in ordinary transverse tests. The effect of time on the strength and stiffness of loaded beams was investigated by Thurston, Kidder and Herman Haupt.

These experiments also show the increased deflection due to time, and the beam broke in every case with a smaller load than would have been required to break it immediately. Thurston's results show that it is not safe to load beams up to 60 per cent. of their breaking weight, and Kidder considers that one-half would be too great for safety.

The following table has been compiled from experiments made by Kirkaldy :—

TABLE XXV.

Description of timber.	Breadth and depth in in.	Span in ft.	Modulus of rupture in pounds per sq. in.	Modulus of elasticity in pounds per sq. in.
Pitch pine	from 11·18 × 11·30 to 13·10 × 13·10	12	7626	—
Dantzig fir	from 10·00 × 12·00 to 13·25 × 14·38	from 8 to 12	4581	—
Do.	from 2·50 × 10·10 to 3·00 × 10·10	10	3726	571,760
Baltic oak	6·40 × 16·00	10	7686	—
Baltic red	from 11·72 × 11·82 to 11·77 × 11·86	12	4890	—
English oak	from 4·55 × 12·00 to 4·58 × 12·00	10	9762	—
St. Petersburg . . .	3·09 × 11·07 3·08 × 11·02	13	8187	2,446,400
St. Petersburg 1st yellow .	from 2·75 × 8·75 to 3·00 × 8·75	10	8556	1,677,500
St. Petersburg 2nd yellow .	from 2·87 × 8·75 to 2·99 × 8·75	10	6918	1,396,700
Archangel	3·00 × 11·06 3·09 × 11·02	13	6738	2,014,300
Archangel Deal, 2A . .	3·00 × 3·00	10	6252	2,043,400
Swedish	from 3·08 × 11·07 to 4·10 × 9·10	10 to 13	5663	1,838,300
Swedish SS	from 3·00 × 9·10 to 3·15 × 9·10	10	6258	1,149,600
Swedish DDD . . .	from 2·93 × 8·75 to 2·95 × 8·75	10	6978	1,528,700

LONGITUDINAL SHEAR.

TABLE XXVI.

RESISTANCE TO SPLITTING DUE TO LONGITUDINAL SHEAR UNDER CROSS-BENDING (HATT).

Kind of timber.	Dimensions.	Sticks failed.				Sticks did not fail.			
		Number.	Shear.			Number.	Shear.		
			Pounds per sq. in.				Pounds per sq. in.		
			Average.	Max.	Min.		Average.	Max.	Min.
Red fir A .	8″ × 16″ × 16′	8	297	345	228	24	170	413	79
	6″ × 8″ × 16′	5	343	517	146				
„ „ B .	8″ × 16″ × 16′	7	223	273	173	11	201	318	117
„ „ C .	8″ × 16″ × 16′ 6″ × 8″ × 16′	7	247	395	74	22	171	390	96
Longleaf pine, Georgia .	10″ × 12″ × 16′	3	268	291	247	19	243	330	151

When a beam becomes weakened by wind shakes or season checks it is apt to fail under longitudinal shear. That is to say, it splits from the ends towards the centre, instead of failing by tension or compression in the centre of the beam. See Chapter V. In beams of short span the resistance to horizontal shear along the neutral axis decides the strength of the beam, not the modulus of rupture. The values given in the above table may be safely used in deciding the ratio of span to depth in beams of similar timber.

The following table gives the results of shearing blocks of timber along the fibre by Hatt, and he states that bending and compression were prevented, and the results are the true shearing strength along the fibre :—

TABLE XXVII.

SHEARING STRENGTH.

Kind of timber.	Dimensions in in.	Grade.	Number of tests.	Number of rings per in.	Moisture per cent.	Ultimate shearing stress, pounds per sq. in.
Red fir A	1·5 × 3	All—				
		Average	104	17·9	19·8	702
		Maximum		34·0	22·1	1136
		Minimum		5·0	17·1	296
Red fir B	1·5 × 3	All—				
		Average	108	11·1	—	720
		Maximum		27·0	—	1025
		Minimum		30·0	—	286
Western hemlock from Oregon	1·5 × 3	All—				
		Average	120	9·9	33·4	764
		Maximum		23·0	53·9	1058
		Minimum		5·0	20·2	496
Loblolly pine from Virginia	1·5 × 3	Square edge—				
		Average	77	3·6	83·0	657
		Maximum		11·0	180·0	1047
		Minimum		2·0	36·0	342
Loblolly pine from South Carolina	1·5 × 3	Square edge, kiln dried—				
		Average	26	5·1	0·7	480
		Maximum		8·5	5·7	779
		Minimum		2·0	0	251
		Soaked—				
		Average	27	4·6	73·6	607
		Maximum		1·0	180·3	941
		Minimum		2·0	40·3	276
Longleaf pine from Georgia	1·5 × 3	Merchantable—				
		Average	38	21·6	21·4	992
		Maximum		39·0	39·4	1257
		Minimum		8·0	18·5	800

The following values for the shearing resistance of timber were obtained at the Watertown Arsenal:—

TABLE XXVIII.

SHEARING STRENGTH OF AMERICAN TIMBER.

Name of timber.	Shearing strength in pounds per sq. in.	
	Minimum.	Maximum.
Ash	458	700
Yellow birch	563	815
White maple	367	647
Red oak	726	999
White oak	752	966
White pine	267	366
Yellow pine	286	415
Spruce	253	374
Whitewood	382	406

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— Green

Transverse Strengt Specimens for of approx: 20 inches sectional area			
No of Tests	Ultimate Extreme Fibre Stress in lbs per sq. inch	Extreme Fibre Stress at apparent Elastic limit in lbs per sq.in.	Modulus of Elasticity in lbs per sq. inch
192	12100	8700	2.450.0
	17.000	15000	3.500.0
128	8.750	6100	1.800.0
	14.500	12800	3.300.0
134	9.400	7.000	1.700.0
	16.700	15600	3.625.0
139	12.000	9000	2.250.0
	17.100	15000	3.200.0
112	6.800	5100	740.0
	10.400	9.600	1.409.0
160	14.600	12.480	2.175.0
	19.400	18.500	3.400.0
147	12.500	10.180	2.000.0
	16.200	15.200	2.700.0
104	13.000	10.800	1.985.0
	17.100	15200	2.550.0
137	11.400	9.300	1.760.0
	16.000	14.900	2.800.0
117	10.425	8.300	1.700.0
	14.800	13.800	2.350.0
124	12.380	9.700	2.050.0
	16.100	15.000	2.800.0
119	11.700	9.800	1.890.0
	16.200	15.500	2.670.0
44	11.250	9.200	1.625.0
	15.700	14.700	2.400.0
62	13.200	11.180	1.950.0
	16.500	14.500	2.700.0
203	10.600	8.300	1.450.0
	15.000	10.300	2.080.0
312	11.500	8.600	1.750.0
	17.300	13.550	2.680.0
[illegible]	11.800	9.300	1.630.0
	17.900	15.900	2.560.0
287	14.000	11.750	1.850.0
	16.100	13650	2.190.0
255	10.500	8.000	1.650.0
	14.200	11.000	2.000.0
252	11.350	9.000	1.800.0
	16.600	12.600	2.590.0
122	16.700	12.700	2.300.0
	21.500	17.000	2.800.0
[illegible]	12.200	10.000	1.500.0
	14.500	11.000	1.800.0
[illegible]	16.300	12.600	2100.0
	20.100	15.000	2.500.0
24	15.600	11.600	2.150.0
	16.900	12.290	2.400.0

Bovey found the average strength of Douglas fir in pounds per sq. in. 412, 377, and 404 according as the plane of shear is tangential, at right angles, or oblique to the annular rings. Bovey states that the shearing strength of timber is much less near the heart than at a distance from the heart. Generally the shearing strength increases with the weight per cub. ft., and with the proportion of hard to soft fibre. The resistance to shearing horizontally along the neutral layer of a beam may be less than the resistances to the direct stresses at the extreme fibres due to bending, and it appears from both Lanza's and Johnson's experiments that, for a distributed load the length of span should be greater than twenty times the depth; and for a concentrated load greater than ten times the depth, otherwise the beams must be dimensioned for shearing. This general rule assumes that the modulus of rupture is about twenty times the resistance to horizontal shearing:—

$$\frac{l}{d} = \frac{f}{s_h} \text{ distributed loads.}$$

$$\frac{l}{d} = \frac{f}{2s_h} \text{ loaded in the centre.}$$

Australian Timber.—The physical properties of Australian timbers are not of such general interest outside Australia as to require, in a work of this kind, anything beyond a brief statement. These timbers are, however, largely exported to other countries and used for railway sleepers, wood paving, and other purposes. In Australia they are also used extensively for engineering and building construction on account of their extraordinary strength and durability. For more complete information on this subject the following reports may be consulted:—

Australian Timbers, by the writer, published by the Government Printer, Sydney, 1892, and Mr. G. A. Julius' report, published by the Government of Western Australia, 1907. An extensive investigation is now in progress in the author's laboratory, the results of which will be published shortly. Table XXIX., prepared by Mr. G. A. Julius, gives in a condensed form the more important properties of Australian timbers in a manner suitable for the purposes of the engineer and architect, and may be taken as the most recent reliable information on the subject.

Seasoning of Timber.—Herman von Schrenk[1] states that water may occur in wood in three conditions: (1) it forms the greater part (over 90 per cent.) of the protoplasmic contents of the living cells; (2) it saturates the walls of all cells; and (3) it entirely, or at least partly, fills the cavities of the lifeless cells, fibres, and vessels. Of 100 pounds of water associated with 100 pounds of dry-wood substance taken from 200 pounds of fresh sap-wood of white pine, about 35 pounds are needed to saturate the cell walls, less than 5 pounds are contained in the living cells, and the remaining 60 pounds partly fill the wood-fibre cavities, and form the sap as ordinarily understood. The wood next the bark contains the most water; different trees, even of the same kind and from the same place, differ as to the amount of water they contain.

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[1] *Seasoning of Timber*, by Herman von Schrenk, published by the Government Printing Office, Washington, 1903.

Seasoning is essentially the more or less rapid evaporation of water from wood. The general opinion that trees contain less water in winter than in summer is probably not always correct. There appears to be an intimate relation between the presence of water in wood and decay, as without water no fungus growth is possible. In dry-rot the growth of the fungus may be possible from the moisture contained in the atmosphere, even if no moisture is contained in the wood, but dry wood will not rot or decay. When green wood is exposed to the sun and air, the moisture evaporates and it becomes dry; also it may shrink considerably, but seasoning implies other changes not so well understood.

When timber is exposed to live steam similar changes take place, the water leaves the wood in the form of steam, while the organic compounds in the walls probably coagulate or disintegrate under the high temperature. Evaporation of water takes place largely through the ends in the direction of the axis of the fibres; the rate depends upon the kind of timber and its shape. Thin boards and beams dry faster than thick ones, sap-wood dries faster than heart-wood, and pine more rapidly than oak.

After wood has reached an air-dry condition, it absorbs water in small quantities after rain, and wood exposed to the weather will absorb and lose water for indefinite periods. When soaked in water seasoned timber absorbs water rapidly. The weakening effect of a given percentage of moisture is the same whether the moisture be the original sap in the tree, or whether it is reabsorbed water after the timber has been thoroughly seasoned; the maximum strength according to Johnson corresponds to 3 or 4 per cent. of moisture. Timber is frequently seasoned by kiln drying to prevent the warping and checking of wood, and only rarely to prevent decay. The effect of hot-air drying as ordinarily conducted does not diminish the strength of timber, according to Johnson, but very high temperatures, as used in the vulcanising process, appear to slightly diminish the strength. In Australian timbers the kiln drying and other processes with steam and hot air, all slightly diminished the strength of the timber.

In most of the European railways the cross-sleepers or ties are seasoned before they are treated chemically; they are piled or stacked with spaces between each sleeper in the open air, protected usually by a roof and allowed to remain from five to twelve months. Kiln drying is used to complete the open-air seasoning before treatment with heavy tar oil. Where time cannot be allowed for seasoning in the above manner seasoning by steaming is sometimes used, and this process is said to coagulate the albuminous substances present in the wood, thus rendering the walls of the wood fibres more permeable. Steamed wood ought to last longer than unseasoned wood, and may be good enough for the supply of partially seasoned timber, but it is at best a makeshift. Seasoning by boiling in creosote oil has been used, but like the steaming process, the timber may be overheated on the outside.

In treating timber with the zinc chloride process, it must be seasoned first in order that the salt will penetrate the wood cells and walls, and it must be seasoned afterwards in order that the salt may crystallise, and be thus less likely to leech out during wet weather. Timber put into water shortly after it is felled, and left in water for a year or more, will be perfectly seasoned after a short subsequent exposure to air. Floating timber down rivers by rafting to wash out the sap has been long practised, and such timber after drying will absorb more preservative, and with greater rapidity.

Timber seasoning, according to Schrenk, is a practical method for increasing the length of life of both untreated and treated timber. At the same time it forms the most important preliminary step to successful chemical treatment. The cost of seasoning is insignificant, while the returns amount to a considerable sum in the end.

With the increased cost and scarcity of timber, every step leading towards a more economic use of our supply ought to receive attention. Green timber should be piled in as open piles as possible as soon as it is cut, and so kept until it is air dry. In the case of railway ties the 7 by 2 form of pile is the best. No timber should be treated until it is air dry.

Timber treated with a preservative dissolved in water should be piled after treatment for several months at least to allow the water passed into the wood with the salt to evaporate. Under no circumstances should timber freshly treated with a water solution be exposed to weathering influences.

Weight per Cubic Foot and Strength.—Johnson states that probably in timbers of uniform structure, strength increases with specific weight, independently of species and genus distinction, *i.e.*, *ceteris paribus*, the heavier wood is the stronger. The author found this to be generally the case in all Australian timbers tested. Johnson gives a relation between strength and weight as apparently true, thus: crushing strength endwise of all timbers excepting oaks, in pounds per sq. in., equals 192 times the dry weight in pounds per cub. ft.

Transverse strength of all timbers excepting oaks, in pounds per sq. in., = 300 times the dry weight in pounds per cub. ft.

Comparative Combustibility of Timbers.—Very little information on this subject exists, but an interesting investigation by Mr. G. Rodney Cherry[1] should be noted. Mr. Cherry experimented with a variety of Australian timbers and tabulated his results for the New South Wales Fire Underwriters Association. The most fire-resisting timber was a kind of box (*Eucalyptus Fletcheri*) to which he assigned the relative number 19. Turpentine, well known for its fire-resisting qualities, is marked 12; iron bark, 5; New Zealand pine, 1. A beam of timber supported at each end, and loaded in the centre with a light pressure controlled by a spring, is subjected to a flame of constant heating power applied in the centre. The mean time necessary to burn through the test-pieces, so that they cannot support the central pressure, was noted in each case. The experiments were made with great care, and repeated many times, and the results appear to the author to be reliable.

THE PRESERVATION OF TIMBER.

The saturation of timber with antiseptic materials has been largely used to preserve timber from decay, but none of these processes are effective in resisting wood-boring insects. The following methods have been used:—

1. Kyanising with bichloride of mercury solution.
2. Burnettising with chloride of zinc solution.
3. The Boucherie process with pyrolignite of iron.
4. The Margery process with sulphate of copper.
5. The Bethell process with creosote oil.
6. The Beer process with a solution of borax.
7. The Powell process.

The effect of all the above methods is to produce coagulation of the albumen which is the perishable part of the timber. A perfect antiseptic should be insoluble in cold water and non-volatile.

Kyanising consists in impregnating the timber with a solution of one pound of bichloride of mercury in four gallons of water. It corrodes iron and is expensive.

Burnettising consists of forcing into the pores of the timber a solution of one part of zinc chloride with ten parts of water. This process is cheaper but less effective than kyanising.

The **Bethell** process has been largely used and consists in saturating the timber with bituminous substances obtained from the distillation of coal tar. The antiseptic element is said to be the carbolic acid. The process has been modified in a variety of ways—in the Seely, Robbins, Leuchs and the Hayford processes in America.

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[1] *Comparative Combustibility of Timbers*, Insurance Institution, New South Wales, Australia.

Powellising.—The essential part of this process consists in boiling wood in a saccharine solution after which it is artificially dried. The wood is placed in a cold solution and the temperature is gradually raised to boiling point and maintained at that temperature for some hours, the length of time depending upon the size and nature of the timber to be treated. The moisture in the wood escapes as steam mixed with air, and carries with it much of the colouring matter in the sap. When the evolution of air and steam ceases, the boiling is discontinued and the solution allowed to cool slowly, and in this process it is absorbed into the wood, penetrating every portion of it, and thus replacing the previously expelled air. The timber may be subsequently dried and seasoned. The saccharine solution has been found an excellent vehicle for conveying other substances into the wood. It may be applied to both hard and soft woods with satisfactory results; it does not weaken the timber, and generally it may be considered as an ideal process for preventing dry-rot and the attacks of insects.

CHAPTER III.

RESULTANT OF ANY NUMBER OF FORCES—GRAPHICAL REPRESENTATION OF MOMENTS OF FORCES—THE METHODS OF DETERMINING THE STRESSES IN STRUCTURES.

THE determination of the stresses in structures such as trusses or girders is based upon the two following principles: (*a*) the principle of the resolution of forces; (*b*) the principle of moments. The former may be stated thus: If any number of forces in the same plane act at a point, or at different points, of a rigid body, and are in equilibrium, the algebraic sum of all their components in any direction is zero. That is to say, the tendency to move the body in one direction is exactly balanced by an equal tendency in the opposite. The moment of a force about a given point may be defined as the product of the force and the perpendicular distance from the point to the line of action of the force. The second principle may then be stated thus: If any number of forces in the same plane act at a point, or at different points, of a rigid body, and are in equilibrium, the algebraic sum of the moments of these forces, taken with reference to any point in their plane, is zero. That is to say, the tendency to produce rotation of the body in one direction is exactly balanced by an equal tendency to produce rotation in the opposite direction. These two fundamental principles give rise to two methods of calculation, and each may be applied analytically or graphically, but it will generally happen in any particular problem that one method offers advantages in simplicity over the others. In order to illustrate these principles, let P_1, P_2, P_3, P_4 represent forces acting in the same vertical plane, θ_1, θ_2, θ_3, θ_4 denote the angles they make with the horizon; let x_1, x_2, x_3, x_4 denote the perpendicular distances from any particular point in the plane of the forces to the lines of action of the forces—that is to say, the moment of a force P_1 about a point O situated at a perpendicular distance x_1 from it is P_1x_1.

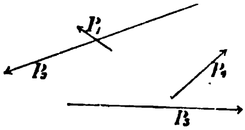

FIG. 42.

Fig. 42 represents four forces in equilibrium, and the lines are assumed to represent the forces in magnitude and direction, and the arrow-heads indicate the sense of each particular force. Now, the first principle may be stated thus:—

$$P_1 \sin\theta_1 + P_2 \sin\theta_2 + P_3 \sin\theta_3 + P_4 \sin\theta_4 = 0, \text{ or } \Sigma Y = 0.$$
$$P_1 \cos\theta_1 + P_2 \cos\theta_2 + P_3 \cos\theta_3 + P_4 \cos\theta_4 = 0, \text{ or } \Sigma X = 0.$$

The second principle may be stated thus:—

$$P_1x_1 + P_2x_2 + P_3x_3 + P_4x_4 = 0, \text{ or } \Sigma Px = 0.$$

The first principle is applied graphically by drawing a polygon, termed the "force polygon," the sides of which are parallel to the forces and equal in length to the magnitude of the forces, as shown in Fig. 43.

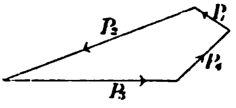

FIG. 43.

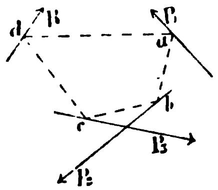

FIG. 44.

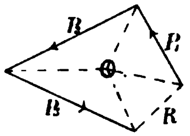

FIG. 45.

This polygon will close, as the forces are in equilibrium. If the polygon does not close, the system of forces is not in equilibrium, and must have a resultant; thus Fig. 44 shows a system of forces not in equilibrium, and Fig. 45 the incomplete polygon.

The closing line, shown dotted, represents the resultant force in magnitude, and is parallel to it in direction; but it does not show its line of action, and, in order to discover this, we proceed as follows:—

Take any point O in Fig. 45 in the plane of the forces, which we will call the pole, and join it to the angular points of the polygon as shown in the dotted lines; draw lines in Fig. 44 parallel to the dotted lines radiating from O in Fig. 45, commencing at any point *a* on the line of action of the force P_1; thus *ab*, Fig. 44, is parallel to the line from O to the join of P_1 and P_2, Fig. 45; *bc* is parallel to the line from O to the join of P_2 and P_3, the points *b* and *c* lying on the forces P_2 and P_3 respectively. To find the point of application of the resultant R, we draw *ad* and *cd* parallel to the remaining lines in Fig. 45, *viz.*, from O to the join of P_3 and R, and from O to the join of P_1 and R. The figure *abcd*, Fig. 44, is termed the "funicular polygon". As the pole O, Fig. 45, may be selected anywhere in the plane, we have an infinite number of funicular polygons all passing through the point *a*, Fig. 44. Again, *a* may be taken anywhere in the line of action of the force P_1, thus producing an infinite number of funicular polygons for the infinite number of positions of *a*, while the pole O, Fig. 45, remains unchanged. But whatever the position of the pole or the starting-point *a*, the intersection *d*, giving the position of the resultant, always falls on the line of action of the resultant. If the system of forces is in equilibrium, the funicular as well as the force polygon must close. The second principle is therefore briefly expressed by stating that for forces in equilibrium the funicular polygon must close.

Culman's Principle.—Suppose we have a single force P acting at A, Fig. 46. The force polygon in this case is represented by the straight line *m*H*n*, where *mn* represents the magnitude of the force to scale, Fig. 47. Select a pole O, and join it with *m* and *n*, which is equivalent to resolving the force P in two directions. Draw through the point A, Fig. 46, two lines parallel to s_1, and s_0, Fig. 47, and draw a line OH perpendicular to *mn*; the distance OH is called the polar distance. Draw *xy* perpendicular to the direction of P, Fig. 46, and draw lines *ab*, *cd*, *ef*, parallel to P. Then the moment of the force P with reference to any point *h* is P × A*h*. But in Fig. 47 we have by similar triangles—

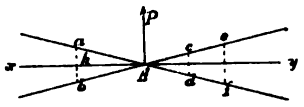

FIG. 46.

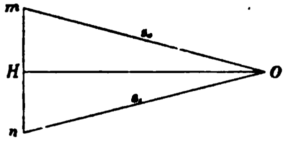

FIG. 47.

$$P : HO :: ab : Ah$$
$$\therefore P \times Ah = HO \times ab.$$

That is, the moment of the force P, with reference to any point, is equal to the ordinate drawn through this point parallel to P, included by the two components into which P is resolved, multiplied by the polar distance in the force polygon.

The same principle may be applied to represent the moments of any number of forces. Let P_1, P_2, P_3 and P_4 represent four forces acting as in Fig. 48. Draw the force polygon *abcde*, Fig. 49, and select a pole O and draw the lines O*a*, O*b*, O*c*, O*d* and O*e*. Draw 1 2, Fig. 48, parallel to O*b*, 2 3 parallel to O*c*, 3 4 parallel to O*d*. Draw

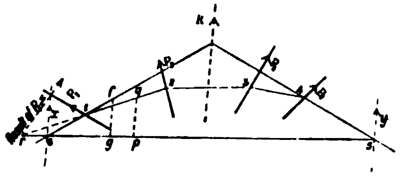

FIG. 48.

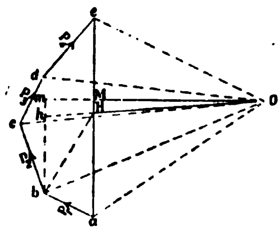

FIG. 49.

a line 6 5 anywhere as a closing line, and OH parallel to it (Fig. 49). Then OH will represent to scale the stress in the closing line 6 5, and aH and He will be the forces at 6 and 5 parallel to the resultant R. Draw OM perpendicular to the resultant R. Let the force x be resolved along 6 1 and 6 5; these are represented in the force polygon by Oa and OH. By Culman's principle the moment of x for any point g is OM × gf, and since OM is constant, the moment is proportional to gf, included between 6 5 and 6 1. The pole distance must be measured to the same scale as the force polygon, the ordinate gf to the scale of distance assumed in Fig. 48.

The ordinate included between 6 1 and 6 5 is proportional to the moment of x for any point through which the ordinate passes.

To find the combined moment of x and P_1 at any point p upon the closing line of all the forces to the left or right of this point, produce 1 2 and 6 5 to meet in r; then the moments of the stresses in 1 2 and 6 5 about the point r will be zero. Hence the moments of P_1 and x about this point must also be zero, since the forces are in equilibrium; but since the combined moment of any number of forces is equal to the moment of the resultant, the moment of the resultant of P_1 and x about r must be zero, also the resultant of P_1 and x must pass through r. The intersection of any two lines in an equilibrium polygon is the point of application for the resultant of the forces at the apices between these lines. Thus the intersection r of 6 5 and 1 2 is the point of application for the resultant of P_1 and x; this is shown in the force polygon by joining Hb. If the force acting at r is resolved along the closing line 6 5 and parallel to the resultant, these components will be represented in the force polygon by bh and hH, where bh is drawn perpendicular to OM; the moment of Hh about p is zero, because it passes through the point p. The moment of bh is equal to the ordinate pq multiplied by the pole distance O upon bm by Culman's principle, *i.e.*, pq × Om, or the projection of Ob in a direction perpendicular to the resultant, or the distance Om. We have, therefore, the following properties of an equilibrium polygon:—

(*a*) The intersection of any two of its lines gives a point upon the resultant of the intermediate forces.

(*b*) The ordinate parallel to the resultant of all the forces multiplied by the projection, on a line perpendicular to this resultant, of the line in the force polygon corresponding to the line in the equilibrium polygon which is cut by this ordinate, gives the combined moment of all the outer forces, right or left, with respect to that point on the closing line through which the ordinate passes.

The ordinates must in all cases be measured to the scale of length, and the projection in the force polygon to the scale of force.

In the case of a beam supported at each end and loaded with any number of loads P_1, P_2, P_3, etc., Fig. 50, draw the force polygon, which is here a straight line, making *ab* equal to P_1, *bc* to P_2, *cd* to P_3, and select a pole O and join O*a*, O*b*, O*c* and O*d*, Fig. 51. In Fig. 50 draw 5 1, 1 2, 2 3 and 3 4 parallel to O*a*, O*b*, etc., in the force polygon and between the directions of reactions V, V_1, and the forces P_1, P_2 and P_3 forming the equilibrium polygon. Join 5 4 and draw OH parallel to it; then *a*H and H*d* represent the reactions V and V_1 respectively. Draw OM perpendicular to the direction of the forces or loads, *i.e.*, horizontal; then the bending moment at any point *p* in the beam of all the forces to the right or left of it is equal to OM × *pq*.

The bending moment may be defined as the bending couple tending to produce rotation of one part of the beam upon the other considered relatively fixed, as at *p*, Fig. 50. It is the resultant moment of the external forces acting on the part of the beam to one side of the section, these moments being taken about a horizontal axis in the section.

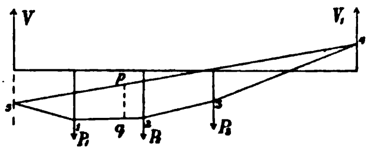

FIG. 50.

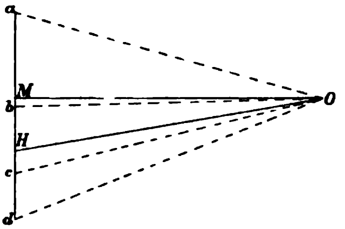

FIG. 51.

The Determination of Stresses in Braced Structures.—The action of vertical loads in producing transverse stresses in beams will be considered in the next chapter, but the principle of moments may also be applied to find the stresses in braced structures. As stated by Rankine: "If a frame be acted upon by any system of external forces, and if that frame be conceived to be completely divided into two parts by an ideal surface, the stresses along the bars which are intersected by that surface balance the external forces which act on each of the two parts of the frame." This theorem furnishes in some cases the most convenient method of determining the stresses along the pieces of a frame. This method may be illustrated with reference to a common roof-truss shown in Fig. 174. The system of lettering adopted is that known as Bow's, which consists in assigning a letter to each enclosed area, and also to each space, enclosed or not, around or bounding the truss, and attaching the same letter to the angle, or point of concourse of lines, which represents the area in the diagram of forces. The angular points are denoted by the numbers 1, 2, 3, . . . 7.

The loads written on Fig. 174, acting vertically downwards, as represented by the arrows, are produced by the dead load of the roof-truss, including its own weight and that of the roof-covering, purlins, ridge, etc., included between two principals, amounting to a total load of 6496 lbs., which is taken as concentrated at the joints 1, 2, 4, 5, . . . 7 in the manner shown in Fig. 174. The resultant upward reaction is therefore 2436 lbs. at each support. Divide the truss by the ideal surface or vertical section *aa*, Fig. 52. This section cuts the two bars 1 2 and 1 3, which, according to our notation, we call AH and AZ respectively. We shall denote the stress in AH or AZ as AH_1 or AZ_1 for shortness. Now, since, before the truss was divided by the section *aa*, the stresses in the two bars cut by it balanced the external forces, it follows that if we apply forces equal to these stresses they also will balance the external forces, and the system will be in equilibrium. The direction in which the arrows point will always be taken away from the portion of the truss which remains after the section is made, and moments which tend to cause rotation in the direction of the hands of a watch will be denoted by the sign + (plus), and moments tending to produce rotation in the opposite direction by the sign − (minus). To find the stress in the bar AZ, we take moments about any point in the bar AH, excepting where it intersects AZ. Select the point 2, and measure the lever arms of the stress AZ_1 and the reaction 2436, which are 2·16 and 10 respectively; these are written on Fig. 52, which represents the forces in equilibrium. We have, therefore—

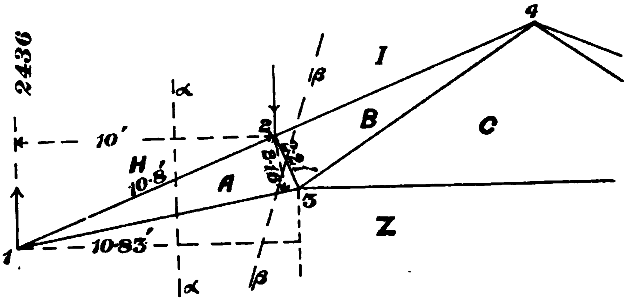

FIG. 52.

$$2436 \times 10 - 2{\cdot}16AZ_1 = 0$$

$$\therefore AZ_1 = +\ 11{,}278 \text{ lbs.}$$

The plus sign shows that the bar is in tension, so that we find both the magnitude and the character of the stress by this method.

To find the stress AH_1, take moments about any point in AZ excepting the point 1; select the point 3 and measure the lever arms; then—

$$2436 \times 10{\cdot}83 + 2{\cdot}21AH_1 = 0, \text{ and } AH_1 = -\ 11{,}938 \text{ lbs.},$$

the minus sign showing that the bar is in compression. In like manner we may proceed to determine the stresses in every member of the truss; thus for the stress AB we take a section $\beta\beta$, which cuts two bars in addition to the bar AB, and in such a case it is most convenient to take moments about the point where these two bars meet, *viz.*, the point 1.

In Fig. 48 we saw that the moments of the stresses in 1 2 and 6 5 were zero, because they were measured about the point *r* where the lines 1 2 and 6 5 intersect. For the same reason the moments of the stress AZ_1 and AH_1 are zero; the moment of the reaction is also zero (Fig. 52);

$$\therefore\ 10{\cdot}8AB_1 + 10 \times 1624 = 0, \text{ and } AB_1 = -\ 1504,$$

which is compression.

To find the stress in BI, take the section $\delta\delta$ and moments about the point 3, Fig. 53.

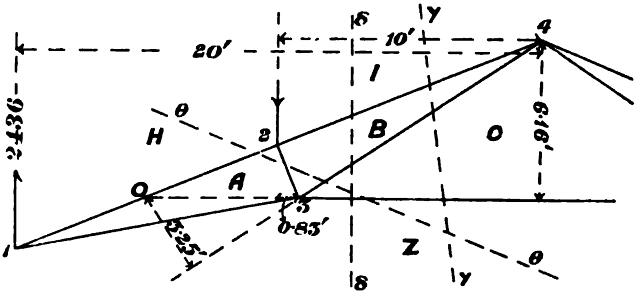

FIG. 53.

$$2{\cdot}21BI_1 - 1624 \times 0{\cdot}83 + 2436 \times 10{\cdot}83 = 0$$

$$\therefore\ BI_1 = -\ 11{,}328 \text{ (compression)}.$$

To find the stress CZ_1, Fig. 53, take the section $\gamma\gamma$ and moments about the point 4—

$$-\ 6{\cdot}16CZ_1 - 1624 \times 10 + 2436 \times 20 = 0$$

$$\therefore\ CZ_1 = 5273 \text{ (tension)}.$$

To find the stress BC_1, take moments about the point O, Fig. 53—

$$-\ 3{\cdot}25BC_1 + 1624 \times 5 + 2436 \times 5 = 0$$

$$\therefore\ BC_1 = 6246 \text{ (tension)}.$$

The right-hand members of the truss will give results corresponding with similar bars on the left, which the student may easily prove. It will be observed that the stress upon any member of the truss is found independently of the stresses in other members. The stress on a particular member can always be found if the section taken does not cut more than three bars. If three or more bars are cut, the stress on any bar can be found if the stress on one or more of the bars cut is known; thus in Fig. 53 the section $\theta\theta$ cuts four bars, and we wish, for example, to find the stress CZ_1. The stress in the bar AB has been found to be -1504.

$$\therefore \; -6{\cdot}16CZ_1 + 2436 \times 20 + (-1504 \times 10{\cdot}8) = 0$$
$$\text{and } CZ_1 = 5270 \text{ (tension) as before.}$$

The method of sections would be applied in the same manner to obtain the stresses in more complicated roof and bridge trusses; thus, if Fig. 54 represents a portion of a bridge truss, the stresses in the three bars cut may obviously be found by taking moments about suitable points; but if the points 3 4 be joined the section cuts four bars, and the method fails. The structure in this case is said to be redundant, as it is indeformable without the line 3 4. If both the bars 3 4 and 2 5 are omitted the structure is deformable, and may alter its shape without altering the length of the members 23, 35, 54 and 42. Generally, if n denote the number of sides of a closed figure, such as 2 3 4 5, and v the number of vertices or corners, then it may be proved that—

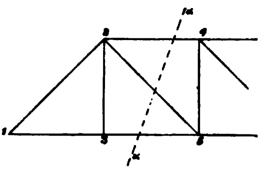

FIG. 54.

For a deformable figure, $n < 2v - 3$.
For an indeformable non-redundant figure, $n = 2v - 3$.
For a redundant figure, $n > 2v - 3$.

Every bridge or roof-truss should be divided up into triangular spaces, so that for every enclosed space bounded by lines $n = 2v - 3$.

The stresses in braced structures, such as the roof-truss illustrated in Fig. 174, may be found graphically by means of a reciprocal figure. Two figures are said to be reciprocal if for every side of the one there is a corresponding side in the other. Corresponding sides are parallel, perpendicular, or inclined at some constant angle to each other. To every system of lines meeting at a point in the one figure there is a corresponding closed polygon in the other. A reciprocal figure can always be drawn if the original figure is indeformable and non-redundant. Take, for example, the roof-truss illustrated in Fig. 174. Draw the force polygon to scale; in this case it becomes a straight line. The convenience of Bow's notation will now be more apparent. The reciprocal figure, Fig. 175, is drawn by commencing at K in the force polygon, and drawing the line KE parallel to the line KE in the roof-truss, Fig. 174, and drawing ZE parallel to ZE in Fig. 174 to meet it in E. Then, since KZ, Fig. 174, represents the reaction at the support 7, which is in equilibrium with the stresses KE_1 and EZ_1, the triangle KEZ, Fig. 175, is the triangle of forces, and the length of the line KE, measured to the same scale as the force polygon, represents the stress KE_1; also the length of the line ZE, Fig. 175, represents the stress ZE_1, and the stress in every other member of the truss is found by completing the reciprocal figure in the manner shown, and measuring the length of each line in Fig. 175 to find the stress in the corresponding member of the roof-truss, Fig. 174. Every line in Fig. 175 is parallel to the corresponding line in Fig. 174, and, completing the figure from D by drawing DE parallel to DE, Fig. 174, and so on, we prove the accuracy of the work by the closing line AH, Fig. 175, being parallel to AH, Fig. 174. It will be noticed that for every space in Fig. 174 there are three lines diverging from a point in Fig. 175 having the same letter against it as the space; thus the space C, Fig. 174, bounded by the lines CB, CD, CZ is represented in Fig. 175 by the three diverging lines radiating from C. The determination of the character of the stresses is second only in importance to the determination of the stresses themselves. In Fig. 175 the thick lines denote the compressive stresses, and the thin lines the tensile, which are determined as follows :—

The forces 2436, KE, and EZ, Fig. 174, are in equilibrium, and are represented by the closed polygon KEZ, Fig. 175. Follow round the polygon, Fig. 175, starting with the known upward reaction ZK = 2436, and indicate the direction of the forces by arrows as shown; put arrows on the corresponding stresses in Fig. 174, then it is seen that KE acts towards the point 7, and is therefore compressive; EZ acts away from it, and is tensile. If EK is produced to the right, it is seen that the stress must be tensile. Hence the following rule: Take any apex of the frame as a system of forces in equilibrium. Follow round the polygon formed by these forces in the direction indicated by those forces whose direction is already known, and transfer the directions thus obtained from the forces to the apex under consideration. If the stress acts towards the apex, the piece is in compression; if away from the apex, the piece is in tension. The results found by measuring the lines to the proper scale in Fig. 175 may now be compared with those obtained by the method of sections. The principle of the resolution of forces may be applied algebraically by using the general equations of equilibrium, and the stresses determined by means of the method of sections and also by means of the reciprocal figure might have been determined by means of these equations. The objection to the algebraic method in such cases is the necessity of measuring all the angles which the bars make with their horizontal and vertical components. The method is too tedious when applied to a single roof or other truss, but may be useful when applied to a number of such trusses of similar angles, but with bars of different length.

CHAPTER IV.

BENDING MOMENTS AND SHEARING STRESSES.

THE action of transverse loads in producing stresses in beams may be conveniently studied with reference to a beam fixed at one end and loaded at the other.

Fig. 55 shows such a beam fixed by building it into a wall, and loaded at the extremity with a load W. The deflection or bending produced by the load is exaggerated for the sake of clearness, and it may be observed that the fibres on the convex side of the beam are lengthened, while those on the concave side are shortened. The upper part of the beam is consequently in tension, and the lower part is in compression, while, somewhere between, tension must change to compression. The layer of fibres at which the change from tension to compression occurs is called the neutral layer, as it is neither lengthened nor shortened by the bending of the beam, and is therefore not subjected to either tensile or compressive stresses.

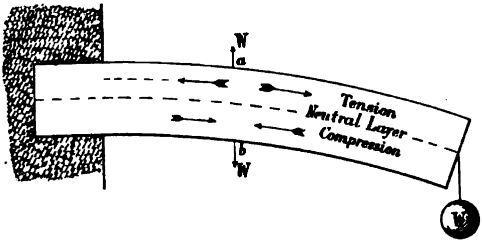

FIG. 55.

If the beam is supported at both ends and loaded in any manner by vertical loads between the points of support, the tensile and compressive stresses developed in the beam will be as indicated in Fig. 56.

Besides the direct tensile and compressive stresses which are developed in the beam in consequence of the bending, there are other stresses which may be realised by clamping together a number of planks, such as would be obtained by making horizontal sections of the beam shown in Fig. 56,

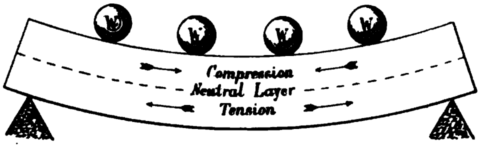

FIG. 56.

and loading in a similar manner. It will then be observed that each section slides upon the other, and overlaps in the manner indicated in Fig. 57; hence we conclude that there are horizontal stresses developed in a beam when loaded which tend to make the various layers slide upon each other. Again, if, in the beam shown in Fig. 55, we imagine two forces applied at any section *ab*, each equal to W, and acting in opposite directions so as to neutralise each other, it is clear that the three forces are equivalent to a couple tending to produce rotation of the beam to the right of the section *ab*, and an unbalanced force W. Hence we conclude that there are vertical stresses developed in a loaded beam which tend to make the layers slide upon each other in vertical planes, as indicated in Figs. 58 and 59. These horizontal and vertical stresses are termed shearing stresses.

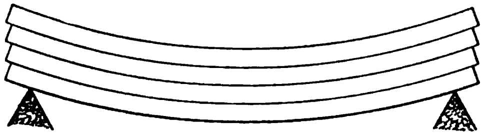

FIG. 57.

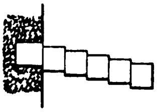

FIG. 58.

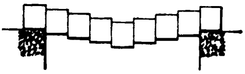

FIG. 59.

The stresses produced in beams subjected to transverse loads may be summarised as follows:—

(*a*) Direct tensile and compressive stresses. (*b*) Horizontal and vertical shearing stresses.

The term "bending moment" has been defined in Chapter III. The bending moments and shearing stresses for the various cases which occur frequently in connection with the construction of ordinary beams and girders will now be considered.

Case I., Figs. 60, 61 and 62.—A beam fixed at one end, termed a cantilever, is loaded at the other with a load denoted by W. This and the following problems may be solved by applying Culman's principle, but it is proposed to treat them algebraically as being rather more convenient for this class of problem. Let x denote the distance from the extremity to the section at which the bending moment is required; let c denote the length of the cantilever. Let M denote the bending moment, and S the shearing stress. From considerations in connection with Fig. 52, it follows that—

$$\mathrm{M} = \mathrm{W}x, \text{ and } \mathrm{S} = \mathrm{W}.$$

M will have its greatest value when $x = c$, and S is constant throughout the length of the beam. Hence the bending moments are represented graphically in Fig. 61, where the maximum bending moment Wc is drawn to scale, and the bending moment at any point may be found by measurement. In a similar manner Fig. 62 represents the constant shearing stress. If W = 10 pounds, and $c = 8$ in., then Wc = 80 inch-pounds, so that the bending moments are expressed as inch-pounds or foot-tons, according to the units of load and distance used. The shearing force is 10 pounds.

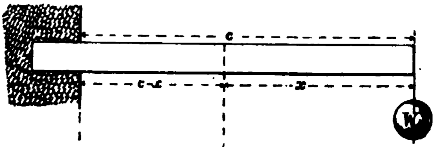

FIG. 60.

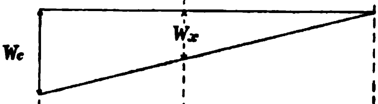

FIG. 61.

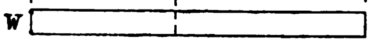

FIG. 62.

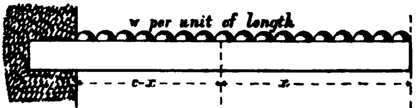

FIG. 63.

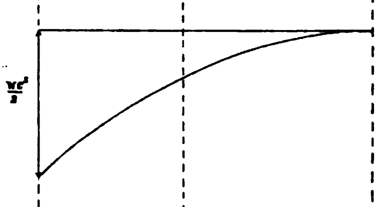

FIG. 64.

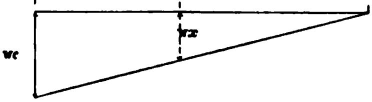

FIG. 65.

Case II., Figs. 63, 64 and 65.—A cantilever loaded with a uniformly distributed load of w per unit of length over its whole length. Adopting the same notation as before, and remembering that the uniform load to the right of the section may be considered as concentrated at its centre of gravity, and acting with a leverage equal to the distance of the centre of gravity from the section under consideration, *viz.*, $\frac{x}{2}$, we have—

$$M = wx \times \frac{x}{2} = \frac{wx^2}{2},$$

which is a maximum when $x = c$, *viz.*, $M = \frac{wc^2}{2}$, at the point of fixing. Again, the shearing stress is equal to the load on the right of the section, which increases uniformly to the point of fixing; hence Fig. 64 represents the bending moments, and Fig. 65 the shearing stresses on the same principles as explained in connection with Case I. It may be observed that in Case I. the equation of bending moments is that of a straight line, while in Case II. the equation $M = \frac{wx^2}{2}$ is that of a parabola, the origin being at the extremity; hence we may determine points in the curve by solving the equation for M after substituting for x its value in ft. or in., and for w its value in tons or pounds; or the curve may be drawn by any of the geometrical methods, having first calculated the maximum ordinate $\frac{wc^2}{2}$.

Let $w = \frac{1}{2}$ a ton per ft. run, and let $c = 6$ ft.; then—

if $x = 1$; $M = 0·25$ foot-tons.
$x = 2$; $M = 1·00$ „ „
$x = 3$; $M = 2·25$ „ „
$x = 4$; $M = 4·00$ „ „
$x = 5$; $M = 6·25$ „ „
$x = 6$; $M = 9·00$ „ „

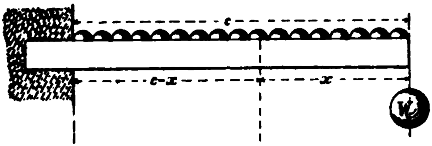

FIG. 66.

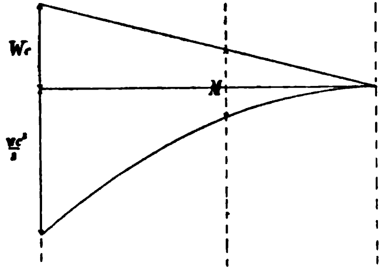

FIG. 67.

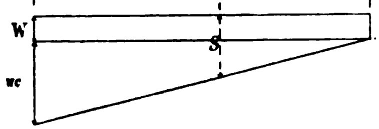

FIG. 68.

Case III., Figs. 66, 67 and 68.—A cantilever loaded at the extremity with a load W, and with a uniformly distributed load of w per unit of length. This case is a combination of Cases I. and II., and the equations may be written down at once, thus—

$$M = Wx + \frac{wx^2}{2}$$

$$S = W + wx.$$

The diagrams of bending moments and shearing stresses are shown in Figs. 67 and 68.

Case IV., Figs. 69, 70 and 71.—A cantilever loaded with any number of concentrated loads.

Let W_1, W_2, W_3, etc., denote the loads, and x_1, x_2, x_3, etc., the distances from the fixed end respectively. Then the moment at b is $M_b = W_3(x_3 - x_2)$; in a similar manner $M_a = W_3(x_3 - x_1) + W_2(x_2 - x_1)$. The moment about the fixed point is—

$$M = W_3x_3 + W_2x_2 + W_1x_1 + \ldots$$

The shearing stress is zero from the extremity to c; from c to b it is W_3; from b to a it is $W_3 + W_2$; from a to the point of fixing it is $W_3 + W_2 + W_1 + \ldots$

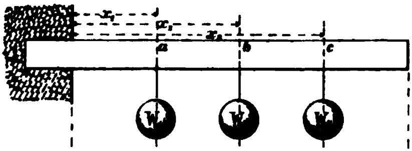

FIG. 69.

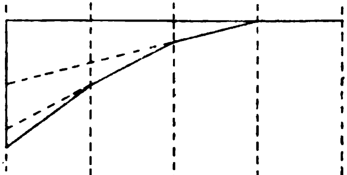

FIG. 70.

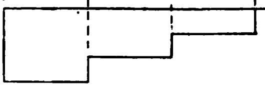

FIG. 71.

Hence the diagrams Figs. 70 and 71 represent the bending moments and shearing stresses respectively.

Case V., Figs. 72, 73 and 74.—A cantilever partially loaded with a uniformly distributed load of w per unit of length (Fig. 72). Let x denote the distance of the section of the beam at which the bending moment is required from the point of fixing; then, considering the

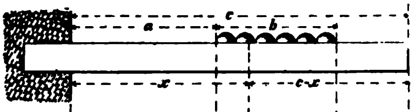

FIG. 72.

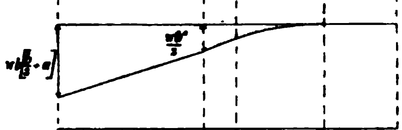

FIG. 73.

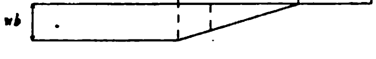

FIG. 74.

loads to the right of the section, we have: Load $= w(a + b - x)$ acting with a leverage $= \dfrac{a + b - x}{2}$; therefore the bending moment is

$$M = \frac{w}{2}(a + b - x)^2.$$

The bending moment at a section situated at a distance a from the point of fixing is found by making $x = a$ in the foregoing equation, thus—

$$M = \frac{w}{2}b^2.$$

To find the bending moment between this last section and the point of fixing, we observe that a is greater than x; thus—

$$M = wb\left(\frac{b}{2} + a - x\right).$$

The maximum bending moment occurs at the point of fixing, where $x = o$; then—

$$M = wb\left(\frac{b}{2} + a\right).$$

We observe that the diagram of bending moments, Fig. 73, is a parabola immediately under the load, and a straight line (*viz.*, the tangent at the extremity of the parabola) from the left extremity of the load to the point of fixing. The diagram of shearing stresses, Fig. 74, needs no comment

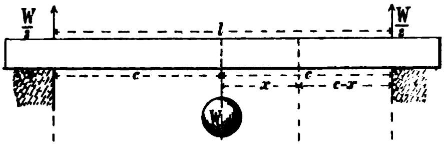

Fig. 75.

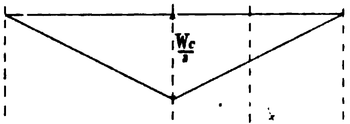

Fig. 76.

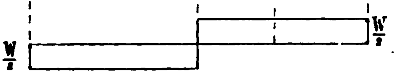

Fig. 77.

Case VI., Figs. 75, 76 and 77.—A beam supported at both ends and loaded in the centre. Let l = the distance between the points of support; c = the half distance; x = the distance of the section at which the bending moment is required from the centre of the beam. Then the pressures on the points of support will be equal to one another, and to half the load W, and there will be a corresponding reaction acting upwards of the same amount. We may consider the beam fixed at the point where the bending moment is required, and consider the forces on one side of it as acting on a beam fixed at one end. Considering the forces to the right of the section, and denoting forces acting upwards by the sign $-$, and those acting downwards by the sign $+$, we have—

$$M = -\frac{W}{2}(c - x),$$

which is a maximum, and $= \frac{Wc}{2}$ when $x = o$, *i.e.*, at the centre of the beam.

If we had taken the moments of forces to the left of the section, we should have found—

$$M = -\frac{W}{2}(c + x) + Wx = -\frac{W}{2}(c - x)$$

as before.

Since $l = 2c$, we may write—

$$M = -\frac{W}{2}\left(\frac{l}{2} - x\right),$$

and when $x = o$, $M = -\frac{Wl}{4}$.

The shearing stress is constant from the supports to the centre, and equals $\frac{W}{2}$; at the centre the shearing stress is zero. The diagrams of bending moments and shearing stresses are shown in Figs. 76 and 77 respectively.

Case VII., Figs. 78, 79 and 80.—A beam supported at both ends, and loaded at a point between the centre and one of the supports (Fig. 78). Let m and n denote the segments into which the load W divides the beam, and let x denote the distance of any section from the right support. The reaction at the left support is, by the law of the lever, $\frac{Wm}{l}$, and at the right support $\frac{Wn}{l}$. Then—

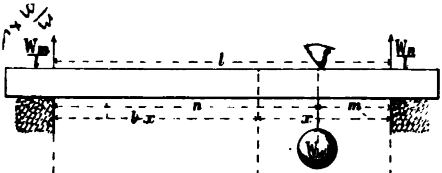

FIG. 78.

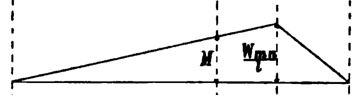

FIG. 79.

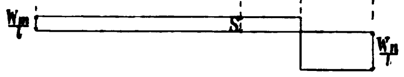

FIG. 80.

$$M = -\frac{Wnx}{l} + W(x - m),$$

which equals a maximum when $x = m$, *i.e.*, at the point of application of the load—

$$M = \frac{Wmn}{l}.$$

The shearing stress is constant from the right abutment to the point of application of the load, and equal to $\frac{Wn}{l}$; it is also constant from this point to the left abutment, and equal $\frac{Wm}{l}$. The diagrams of bending moments and shearing stresses are shown in Figs. 79 and 80 respectively.

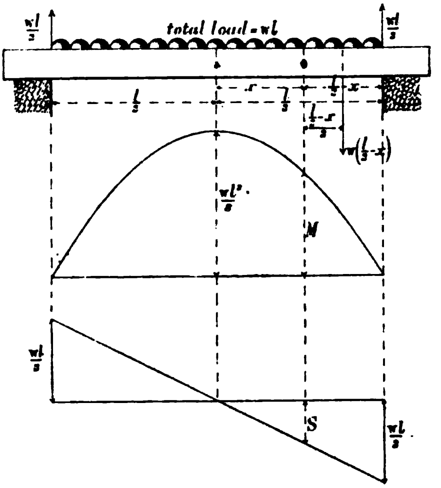

FIG. 81.

FIG. 82.

FIG. 83.

Case VIII., Figs. 81, 82 and 83.—A beam supported at both ends, and loaded with a uniformly distributed load of w per unit of length, Fig. 81. The reactions at the supports are both equal to half the load, *i.e.*, $\frac{wl}{2}$. Considering the distributed load to the left or right of the section as concentrated at its centre of gravity, and acting with a leverage equal to the distance of this point from the section under consideration as in Case II, we have, denoting the distance of any section from the centre by x—

$$M = -\frac{wl}{2}\left(\frac{l}{2}-x\right)+\frac{w}{2}\left(\frac{l}{2}-x\right)^2 = -\frac{w}{2}\left\{\left(\frac{l}{2}\right)^2-x^2\right\}.$$

M is a maximum when $x = o$; *i.e.*, at the centre of the beam we have $M = -\frac{wl^2}{8}$. If x is measured from one of the points of support, we have $M = -\frac{w}{2}(lx - x^2)$. We observe that the curve of bending moments is a parabola (Fig. 82), the central ordinate of which is equal to $\frac{wl^2}{8}$; the bending moment at any other point may be found by drawing the parabola and measuring the particular ordinate to the same scale as $\frac{wl^2}{8}$, or we may make use of the equation of bending moments, substituting for x its distance from the centre of the beam, or from a support.

If the beam or girder is 60 feet between the centres of the supports, and the load per foot run is 1·13 tons, we have the equation of bending moments $M = \frac{1\cdot13}{2}(30^2 - x^2)$, where x is measured from the centre, and we may tabulate the results as in Fig. 84.

The shearing stress at any section in the beam is found by subtracting the load between the section and the nearer support from the reaction at that support. Since the reaction at both supports is half the load, *viz.*, $\frac{wl}{2}$, it follows that the shearing stress must gradually diminish towards the centre of the beam, where it is zero. The shearing-stress diagram is drawn in Fig. 83. If we measure x from the centre of the beam, $S = wx$.

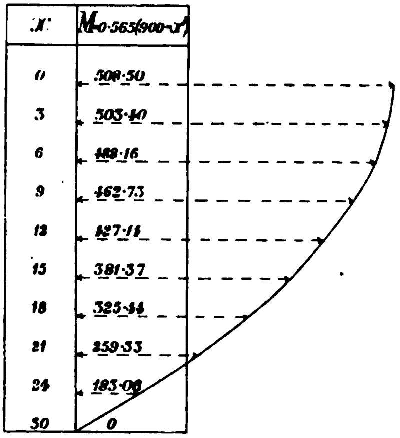

FIG. 84.

Case IX.—A beam supported at both ends and loaded with a central load W, also with a uniformly distributed load of w per unit of length. This is obviously a combination of Cases VI. and VIII., and the equations of bending moments and shearing stress may be written thus—

$$M = -\frac{W}{2}\left(\frac{l}{2} - x\right) - \frac{w}{2}\left\{\left(\frac{l}{2}\right) - x^2\right\}.$$

When $x = o$, M is a maximum, and

$$M = -\frac{Wl}{4} - \frac{wl^2}{8}.$$

$$\text{Also } S = \frac{W}{2} + wx,$$

which has its maximum value when $x = \frac{l}{2}$, *i.e.*,

$$S = \frac{W}{2} + \frac{wl}{2}.$$

Case X., Figs. 85, 86 and 87.—A beam supported at both ends, and loaded with a partially distributed load of w per unit of length (Fig. 85). The total load is wb, and the load to the left of the section at a distance x from the left abutment is $w\,(x - a)$, acting with a leverage of $\frac{x - a}{2}$. The reaction at the right support is $\frac{wb}{l}\left(a + \frac{b}{2}\right)$, and at the left support it is $\frac{wb}{l}\left\{l - \left(a + \frac{b}{2}\right)\right\}$. Taking moments to the left of the section under consideration, we have—

$$M = -\frac{wbx}{l}\left\{l - \left(a + \frac{b}{2}\right)\right\} + \frac{w}{2}(x - a)^2.$$

The maximum value of M is found by making $\frac{dM}{dx} = o$, or

$$- \frac{wb}{l}\left\{l - \left(a + \frac{b}{2}\right)\right\} + w(x - a) = o.$$

$$\therefore x = a + b - b\left(\frac{2a + b}{2l}\right).$$

$$\text{If } x = a,\ \text{M} = - \frac{wab}{l}\left\{l - \frac{2a + b}{2}\right\}.$$

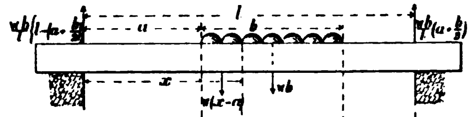
$\frac{wb}{l}\left\{l-\left(a+\frac{b}{2}\right)\right\}$
l
a
b
$\frac{wb}{l}\left(a+\frac{b}{2}\right)$
x
$w(x-a)$
wb

FIG. 85.

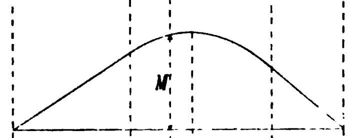

FIG. 86.

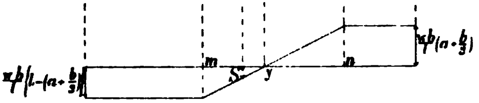
$\frac{wb}{l}\left(a+\frac{b}{2}\right)$
m
n
$\frac{wb}{l}\left\{l-\left(a+\frac{b}{2}\right)\right\}$
S
y

FIG. 87.

If x falls between the left of the load and the left support—

$$\text{M} = - \frac{wbx}{l}\left(l - \frac{2a + b}{2}\right).$$

In a similar manner, if x falls between the right of the load and the right support—

$$\text{M} = - \frac{wbx}{l}\left(a + \frac{b}{2}\right).$$

The diagram of bending moments, Fig. 86, is a parabola for the length b immediately under the load, and straight lines tangential to the parabola between its extremities and the supports.

The shearing stress from the right support to the right extremity of the load is equal to the reaction; thus—

$$\text{S} = - \frac{wb}{l}\left(a + \frac{b}{2}\right).$$

In a similar manner, the shearing stress from the left support to the extreme left of the load is—

$$\text{S} = - \frac{wb}{l}\left\{l - \left(a + \frac{b}{2}\right)\right\}.$$

At any section intersecting the load the shearing stress is—

$$\text{S} = - \frac{wb}{l}\left\{l - \left(a + \frac{b}{2}\right)\right\} + w(x - a).$$

It equals zero for some section intersecting the partial load, which section may be found by making the above expression for S equal to zero. The maximum value occurs when $x = a$, or $x = a + b$, in which cases S is the reaction at the left and right supports respectively. The diagram of shearing stresses is illustrated in Fig. 87, and the point y denotes the position of the section where the shearing stress vanishes, and the bending moment is a maximum.

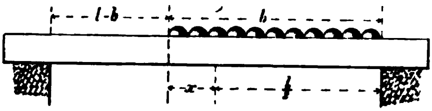

FIG. 88.

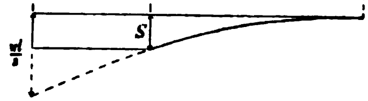

FIG. 89.

If the partial load extends from the right abutment to some point in the beam, then (Fig. 88) $a = l - b$, and the equation of shearing stress becomes—

$$S = -\frac{wb}{l}\left\{l - \left(l - b + \frac{b}{2}\right)\right\} + w\{x - (l - b)\}$$

$$= -\frac{wb^2}{2l} + w\{x - (l - b)\}$$

which has its maximum value when $x = l - b$, or at the extremity of the load, thus—

$$S = -\frac{wb^2}{2l}.$$

This is the equation of a parabola (Fig. 89). It is convenient to denote the distance to which the load extends beyond the centre of the beam, measuring from the right, by x; then $x + \frac{l}{2} = b$, and the equation of maximum shearing stresses becomes—

$$S = -\frac{w}{2l}\left(x + \frac{l}{2}\right)^2.$$

This is in a convenient form for calculating the maximum shearing stress produced by a live load extending from the right support. A similar parabola would indicate the maximum shearing stresses if the live load came on from the left support; the maximum ordinate would, however, in this case appear at the right support. It is clearly only necessary to determine the

x	Value of ·38 x	Value of constant	Total value of S in tons
0	0	5·63	5·63
6	2·28	8·10	10·38
12	4·56	11·02	15·58
18	6·84	14·40	21·24
24	9·12	18·22	27·34
30	11·40	22·50	33·90

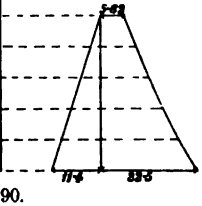

FIG. 90.

values of S for the left half of the beam. As an example, let l = 60 feet, and the live load w = 0·75 tons per foot run. Let the dead load per foot run be 0·38 tons; then the total shearing stress is—

$$S = 0{\cdot}38x + \frac{0{\cdot}75}{120}(x + 30)^2 = 0{\cdot}38x + 0{\cdot}00625(x + 30)^2.$$

The diagram may be drawn as in Fig. 90.

Case XI., Figs. 91, 92 and 93.—A beam supported at both ends and loaded with a series of concentrated loads, W_1, W_2, W_3, etc. (Fig. 91). Let R_1 and R_2 denote the reactions at the supports; let a_1, a_2, a_3, etc., denote the distances of the points of application of W_1, W_2, W_3, etc., respectively measured from the left support. Then $R_2 = \frac{a_1}{l}W_1 + \frac{a_2}{l}W_2 + \frac{a_3}{l}W_3$, and $R_1 + R_2 = W_1 + W_2 + W_3$; hence $R_1 = W_1 + W_2 + W_3 - R_2$.

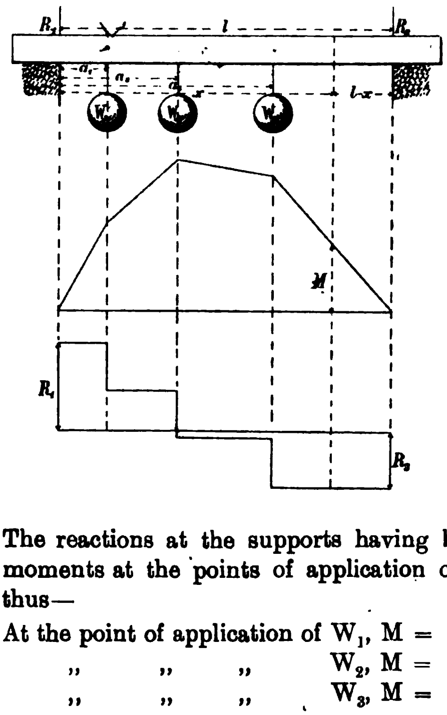

FIG. 91.

FIG. 92.

FIG. 93.

The reactions at the supports having been determined, the bending moments at the points of application of the loads may be expressed thus—

At the point of application of W_1, $M = -R_1a_1$

" " " W_2, $M = -R_1a_2 + W_1(a_2 - a_1)$

" " " W_3, $M = -R_1a_3 + W_2(a_3 - a_2) + W_1(a_3 - a_1)$

At a section distant x from the left support—

$$M = -R_1x + W_1(x - a_1) + W_2(x - a_2) + W_3(x - a_3).$$

The diagram of bending moments, Fig. 92, is an irregular polygon.

The shearing stress from the left support to the point of application of W_1 is R_1; from the point of application of W_1 to that of W_2 it is $(R_1 - W_1)$.

From W_2 to W_3, $S = R_1 - W_1 - W_2$.

From W_3 to the right support, $S = R_1 - W_1 - W_2 - W_3 = R_2$.

The diagram of shearing stresses is therefore a stepped figure (see Fig. 93), As an example of this case, take a beam of 24 ft. span supported at each end, and loaded, as shown in Fig. 94, with four equal weights of 9 tons each. This example corresponds with that of a cross-girder in a railway bridge carrying a double line of way, and receiving its maximum load from the weights concentrated by the driving-wheels of the locomotive.

Since the loads are symmetrically arranged with regard to the centre of the girder, the reactions at the supports are each equal to 18 tons. The bending moment at *a* and *d* is—

$$M = 18 \times 4 = 72 \text{ foot-tons.}$$

$$\text{At } b \text{ and } c,\ M = 18 \times 9 - 9 \times 5 = 117 \text{ foot-tons.}$$

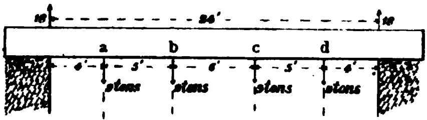

FIG. 94.

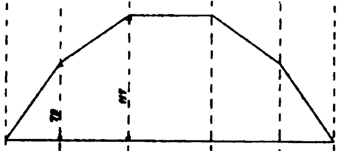

FIG. 95.

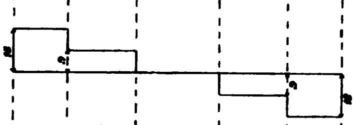

FIG. 96.

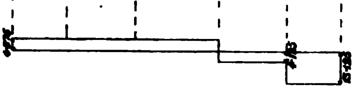

FIG. 97.

The diagram of bending moments is shown in Fig. 95. The shearing-stress diagram, Fig. 96, shows that there is no shearing stress from *b* to *c* in the centre of the girder. If the shearing-stress diagram is drawn for only two loads, as in Fig. 97, the shearing stress in the centre is 4·875 tons, showing that the maximum shearing stress in the centre, *i.e.*, from *b* to *c*, Figs. 94 and 97, of a cross-girder carrying a double line of railway occurs when one line only is traversed by the live load. The maximum shearing stress at other portions of the cross-girder occurs when it is fully loaded on both pairs of rails.

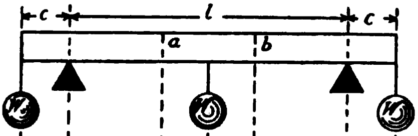

FIG. 98.

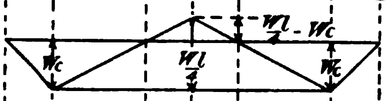

FIG. 99.

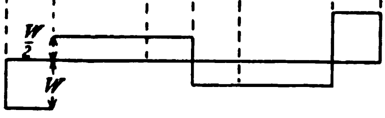

FIG. 100.

Case XII., Figs. 98, 99 and 100.—A beam supported at two points situated at a distance *c* within the extremities of the beam, with equal overhanging portions, and loaded at the centre and ends with concentrated loads each equal to W.

The bending moment at each support is Wc, as in an ordinary cantilever. This downward bending moment produces an upward bending moment between the supports, which is uniform throughout the length l, and also equal to Wc. The central load produces a downward bending moment, which in the centre of the beam is $\frac{Wl}{4}$, so that the actual bending moment in the centre is—

$$M = -Wc + \frac{Wl}{4}.$$

The shearing stress is uniform throughout the length l, and is equal to $\frac{W}{2}$; and throughout each of the overhanging portions, as in the ordinary cantilever, the shearing stress is W; hence the pressure on each support is $W + \frac{W}{2} = \frac{3W}{2}$.

The diagrams of bending moments and shearing stresses are therefore as shown in Figs. 99 and 100, from which it may be observed that at the points a and b there is no bending moment.

Case XIII., Figs. 101, 102 and 103.—A beam supported in a similar manner to that described in Case XII., but loaded with a

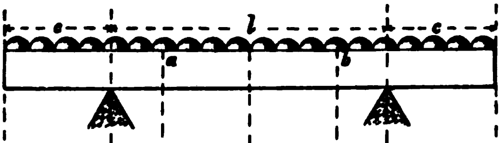

FIG. 101.

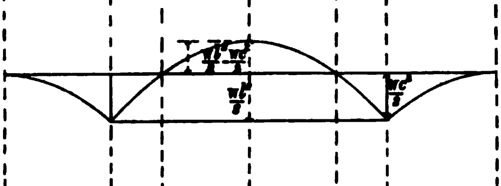

FIG. 102.

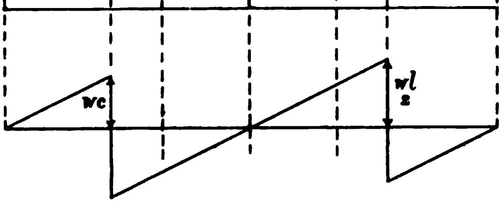

FIG. 103.

uniformly distributed load of w per unit of length (Fig. 101). The bending moment at each support is $\frac{wc^2}{2}$, as in an ordinary cantilever loaded uniformly; this downward bending moment produces a uniform upward bending moment between the supports, which also equals $\frac{wc^2}{2}$.

The load between the supports produces a downward bending moment, which at the centre of the beam is $\frac{wl^2}{8}$, so that the bending moment in the centre of the girder is—

$$M = -\frac{wc^2}{2} + \frac{wl}{8}.$$

The shearing stress for the cantilevers is wc at supports, and for the length l is $\frac{wl}{2}$, therefore the pressure on each support is—

$$wc + \frac{wl}{2}.$$

The diagrams of bending moments and shearing stresses are as shown in Figs. 102 and 103. It may be observed here, as in Case XII., that at a and b there is no bending moment; these points are known as points of inflection or contra-flexure.

Case XIV., Figs. 104, 105 and 106.—A beam supported at two points situated at a distance l from each other, with an overhanging portion. The beam is loaded with a uniformly distributed load of w per unit of length over the distance l, and with a load W concentrated at the extremity of the overhanging portion.

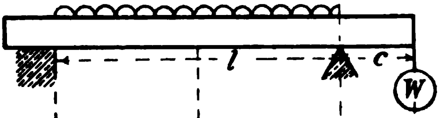

FIG. 104.

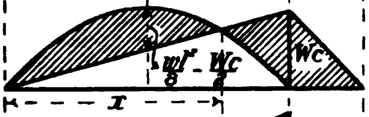

FIG. 105.

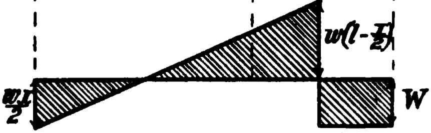

FIG. 106.

The bending moment at the end support is zero, and at the intermediate support Wc. The latter bending moment produces an upward bending, and it diminishes from the support where it is a maximum, to the end support where it is zero.

The uniform load produces a downward bending moment, which is $\frac{wl^2}{8}$ at the centre of the portion l; the resultant bending moment at this point is :—

$$M = \frac{wl^2}{8} - \frac{Wc}{2} \quad \text{(Fig. 105).}$$

If x denotes the distance of the point of contraflexure from the end support, the shearing stresses are as represented in Fig. 106. The pressure on the intermediate support is—

$$W + w\left(l - \frac{x}{2}\right).$$

The general expression for a beam supported at each end and loaded uniformly, is as follows:—

Let x denote the distance from the left point of support considered as origin, then the shearing stress may be written:—

$$S = R - \int_0^x w dx = \frac{wl}{2} - wx = w\left(\frac{l}{2} - x\right).$$

The bending moment at the same section is—

$$M = \int_0^x S dx = w\int_0^x \left(\frac{l}{2} - x\right)dx = \frac{w}{2}(lx - x^2).$$

Again, $\frac{dM}{dx} = \frac{w}{2}(l - 2x) = S.$

The maximum bending moment is found by putting $\frac{dM}{dx} = 0$; or $x = \frac{l}{2}.$

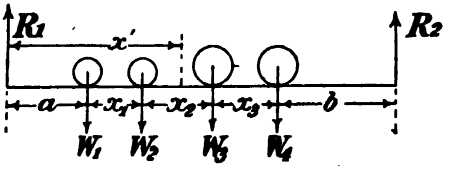

FIG. 107.

Beam supported at each end and loaded with a series of concentrated loads, such as the wheels of a locomotive engine (Fig. 107).

$$R_1 = \frac{1}{l}[W_1(l - a) + W_2\{l - (a + x_1)\} + W_3\{l - (a + x_1 + x_2)\}$$
$$+ W_4\{l - (a + x_1 + x_2 + x_3)\}].$$

The shearing stress at any section situated at a distance x' from the left support is—

$$S = R_1 - (W_1 + W_2), \text{ or } = R_2 - (W_3 + W_4).$$

The bending moment at the same section is—

$$M = R_1x' - \{W_1(x' - a) + W_2(x' - a - x_1)\}$$
$$\text{or } M = R_2(l - x') - [W_3\{(l - x') - (b + x_3)\} + W_4\{(l - x') - b\}].$$

Example.—A beam loaded with a uniformly distributed load of

0·4 tons per foot run, on a span of 30 ft., also with four concentrated loads of $W_1 = W_2 = 6$ tons, and $W_3 = W_4 = 10$ tons, where $a = 2$ ft., $x_1 = 8$ ft., $x_2 = 6$ ft. and $x_3 = 7$ ft. Find the shearing stresses and bending moments at the points of application of the concentrated loads, Fig. 108.

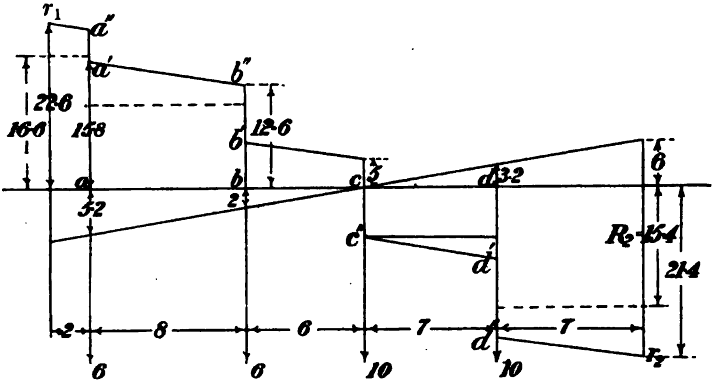

FIG. 108.

For the concentrated loads we have :—

$R_1 = \frac{1}{30}\{6(28+20)+10(14+7)\} \quad . \quad . \quad . \quad . \quad . \quad = 16{\cdot}6.$

$R_2 = 32 - 16{\cdot}6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad = 15{\cdot}4.$

$S_a = 16{\cdot}6 - 6 = 10{\cdot}6\,;\ M_a = 16{\cdot}6 \times 2 \quad . \quad . \quad . \quad = 33{\cdot}2.$

$S_b = 16{\cdot}6 - 12 = 4{\cdot}6\,;\ M_b = 16{\cdot}6 \times 10 - 6 \times 8 \quad . \quad . \quad = 118.$

$S_c = 16{\cdot}6 - 22 = -5{\cdot}4\,;\ M_c = 16{\cdot}6 \times 16 - 6(14+6) \quad . \quad = 145{\cdot}6.$

$S_d = 16{\cdot}6 - 32 = -15{\cdot}4\,;\ M_d = 16{\cdot}6 \times 23 - 6(21+13) - 10 \times 7 = 107{\cdot}8.$

For the distributed load we have :—

$$\text{Reactions at points of support} = \frac{wl}{2} = 6.$$

The shearing stress at $a = 5{\cdot}2.$
" " " $b = 2{\cdot}0.$
" " " $c = -0{\cdot}4.$
" " " $d = -3{\cdot}2.$

$$\text{The bending moment in the centre} = \frac{wl^2}{8} = 45.$$

The shearing stress diagrams for the combined concentrated and dead loads is shown in Fig. 108 by the lines $r_1a''a'b''b'c'c''d'd''r_2$.

The bending moments at a, b, c and d for the concentrated loads, also for the distributed load calculated from the equation—

$$M = \frac{w}{2}(lx - x^2),$$

are shown on the diagram, Fig. 109.

The total bending moments for the combined loads are 44·4, 158, 190·2 and 140·0 at the points *a*, *b*, *c* and *d* respectively.

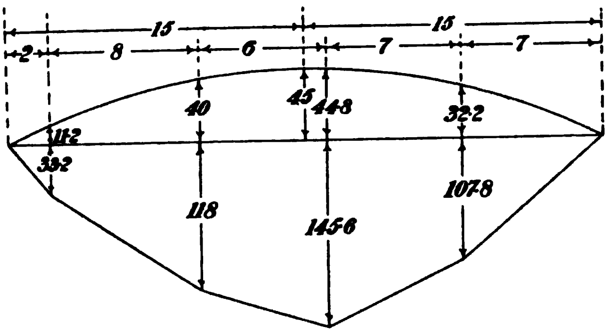

Fig. 109.

The examples given for concentrated and distributed loads, sufficiently illustrate the method of calculating the bending moments, and shearing stresses in the ordinary manner. Other examples will be given in connection with "influence lines".

Influence Lines.—An influence line is a curve or straight line drawn in such a manner that its ordinate at any point represents the moment, shear, panel load, stress, or any similar function due to a unit load moving over the structure. If y denote the stress at any point of a structure at a distance of x from the point of support, then—

$$y = f(x)$$

where y equals some function of x.

The subject of influence lines has been treated by Profs. G. H. Swain[1] and Johnson,[2] by Prof. Burr and Dr. Falk[3]; also by Mr. R. Hawken.[4] The investigation of stresses in connection with the design of structures of all kinds is frequently much simplified by the use of influence lines, as they work out the required function once for all for the structure under consideration; whereas the ordinary methods cover more or less the same ground many times.

It is proposed to briefly consider the subject of influence lines.

Influence line for the shear and the bending moment in a beam or truss (Figs. 110-114).

Let P denote the point at which the shear and the bending moment are required (Fig. 110).

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[1] *Trans. Am. Soc. C. E.*, vol. xvii.

[2] *Framed Structures.*

[3] *The Graphic Method by Influence Lines*, pub. by J. Wiley & Sons.

[4] "Influence Lines," by R. Hawken, B.A., B.E., *Journal of the Sydney University Engineering Society*, 1903.

Let μ denote the ratio of the distance of the point P from the left support to the total length of the span—

$$\mu l = \text{AP}.$$

Let kl denote the distance of the unit load from A, and let

$$W_1 = W_2 = W_3 = 1.$$

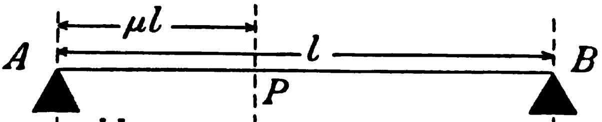

FIG. 110.

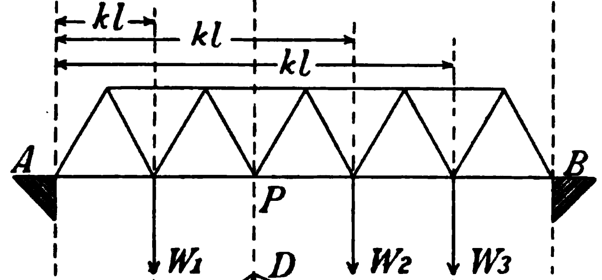

FIG. 111.

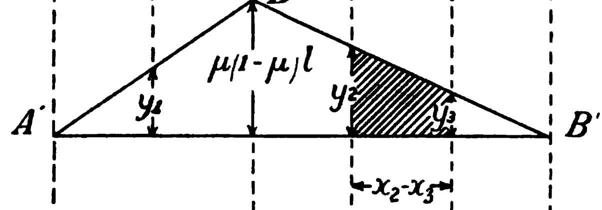

FIG. 112.

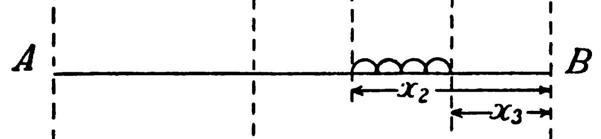

FIG. 113.

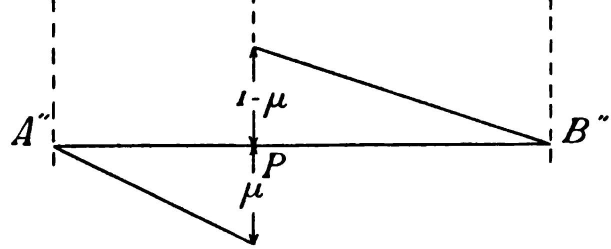

FIG. 114.

The moment at P for a load W_1 on the left of P is—

$$\mu l(1 - k).$$

This is the equation of the straight line A'D (Fig. 112). If the load is at W_2 or W_3 on the right of P, the moment at P is represented by the influence line B′D. The influence line for the moment at P, for unity load situated anywhere between A and B, is represented by A′DB′ (Fig. 112), where DP = $\mu(1 - \mu)l$. The moment at P due to any load W_2 or W_3 is the ordinate y_2 or y_3 multiplied by the load, since the ordinates y_2 and y_3 are equal to unity load placed at these points respectively. The moment at P due to a uniformly distributed load of w per unit of length extending from x_2 to x_3 (Fig. 113) is equal to the area between the extreme ordinates y_2 and y_3 multiplied by w, since—

The moment due to an element wdx of load = $wydx$ where y is the ordinate under the load, and the total moment is—

$$w\int_{x_3}^{x_2} ydx = w \times \text{area shaded (Fig. 112).}$$

The moment for the uniform load extending over the whole length of the span is—

$$w \times \text{area of A}'\text{DB}'.$$

$$\text{The maximum moment} = \frac{w\mu}{2}(1 - \mu)l^2.$$

The maximum moment for a single concentrated load W occurs when the load is applied at P and equals—

$$W\mu(1 - \mu)l.$$

The shear at P in the beam AB due to a load equal to unity moving from B towards A increases from zero for the load at B to + $(1 - \mu)$ for the load just to the right of P. As the load passes P the shear becomes $- \mu$, and decreases to zero at A (Fig. 114). Any movement of loads to the left increases the positive shear at P until some load W passes P, when the shear is suddenly decreased by an amount equal to—

$$W\{(1 - \mu) + \mu\} = W.$$

For concentrated loads the shear reaches a maximum each time a load reaches P. The greatest of these maxima values occurs in a series of moving concentrated loads, such as the wheel loads of a locomotive, when one of the wheels near the head of the train is just to the right of P.

To find the Position of a Series of Concentrated Loads producing the Maximum Moment.—Let a series of wheel loads be spaced as shown in Fig. 115, and let λ_r denote the distance from P to B, and λ_l the distance from P to A. Let ΣW_r denote the load on λ_r, and ΣW_l denote the load on λ_l, then the influence line for the bending moment at P produced by any loads between A and B is ADB, where DP = $\mu(1 - \mu)l$.

The bending moment at P for any load between P and B is equal to the ordinate drawn through the load intercepted by the lines DB and PB multiplied by the load. The ordinate for the wheel W_{r_4} is—

$$\frac{x_4}{\lambda_r}(1 - \mu)\mu l,$$

and the moment at P is—

$$W_{r_4}\frac{x_4}{\lambda_r}(1 - \mu)\mu l.$$

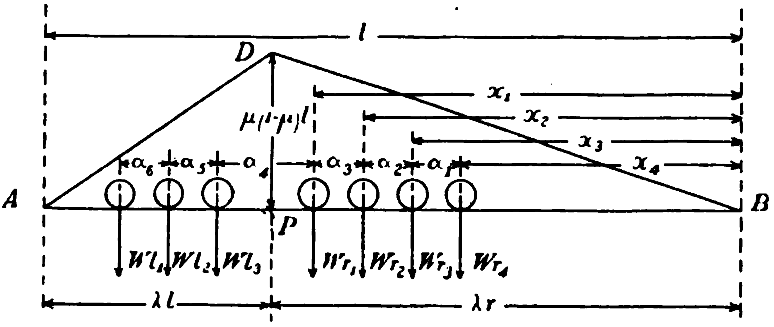

FIG. 115.

For any load between A and P at a distance x from B the ordinate is—

$$\frac{l - x}{\lambda_l}(1 - \mu)\mu l,$$

and the moment at P for the wheel load W_l is—

$$W_l\frac{l - x}{\lambda_l}(1 - \mu)\mu l.$$

The total bending moment for all the loads between A and B may be expressed thus—

$$\Sigma W_r\frac{x_r}{\lambda_r}(1 - \mu)\mu l + \Sigma W_l\frac{l - x_r}{\lambda_l}(1 - \mu)\mu l = M.$$

For a maximum $\frac{dM}{dx} = 0$, and $x_r = x_4 +$ constants, $\therefore \frac{dx_r}{dx} = 1$.

$\therefore$ for the position of the loads producing the maximum moment at P—

$$\Sigma\frac{W_r}{\lambda_r} - \Sigma\frac{W_l}{\lambda_l} = 0;$$

$$\therefore \Sigma\frac{W_r}{W_l} = \frac{\lambda_r}{\lambda_l}.$$

Or when the sum of the loads to the right of P bears the same ratio to the sum of the loads to the left of P, as the distance of P from the right support bears to the distance of P from the left support, the relationship must be as nearly as possible correct when one of the wheel loads is over the point P; then if the loads move slightly to the right or slightly to the left, the ratio is—

$$\frac{\Sigma W_r}{\Sigma W_l} > \text{ or } < \frac{\lambda_r}{\lambda_l}.$$

It also follows that when—

$$\frac{\Sigma W_r}{\Sigma W_l} = \frac{\lambda_r}{\lambda_l}, \text{ then } \frac{\Sigma W_r}{\Sigma W_l + \Sigma W_r} = \frac{\lambda_r}{\lambda_l + \lambda_r}.$$

If we denote the segment λ_r by l' and the load on it W_r by P, also the whole load on the span l by W, we may write—

$$\frac{P}{W} = \frac{l'}{l}.$$

This criterion may be used to determine the position of the wheel loads, on any girder or truss, producing the maximum moment at a given point. Since at the point where the maximum moment occurs the shearing stress is zero, we may write—

$$\frac{l'}{l} = \frac{R}{W}.$$

Or the distance between the point where the maximum moment occurs, and the centre of gravity of the load is bisected by the centre of the span.

The Maximum Load upon a Cross-girder.—The maximum load on a cross-girder in the deck of a bridge is due to the maximum reactions of the stringers or longitudinal girders attached to the cross-girder. Let aP and Pb, Fig. 116, denote the longitudinal girders

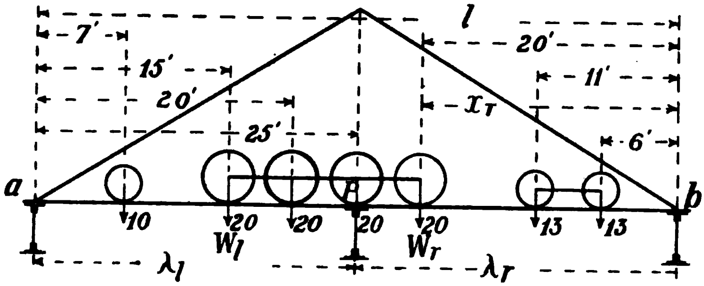

FIG. 116.

which transfer their wheel loads to the cross-girder at P. To make the case general we will assume that the spans of the stringers are not equal.

$$\text{The reaction at } P = \Sigma W_r \frac{x_r}{\lambda_r} + \Sigma W_l \frac{l - x_r}{\lambda_l},$$

since $x_r = x +$ constants, $\frac{dx_r}{dx} = 1$.

For the maximum reaction, $\frac{dP}{dx} = 0$,

$$\therefore \frac{\Sigma W_r}{\Sigma W_l} = \frac{\lambda_r}{\lambda_l}.$$

If the spans are equal, $\Sigma W_r = \Sigma W_l$. Take an example of the wheel loads of Cooper's[1] E_{40}, and place the fourth wheel over P, then it will be seen that—

The loads to the right = 46.
" " left = 50.

Adding the load at P to the loads on the right the result is greater than the loads on the left. This follows from the fact that at the maximum point $\frac{dM}{dx}$ changes sign.

The actual reaction at P due to the loads on the right is 24·84 units, and to the loads on the left 30·8 units. The total maximum reaction is 24·84 + 30·8 + 20 = 75·64 units. The last two investigations are due to Mr. Hawken.

Maximum Shear.—The maximum shear at any point occurs when the beam is loaded from one of the points of support to that point. We will consider the influence line and the position of the loads producing the maximum shear at any point of a truss (Figs. 117 and 118).

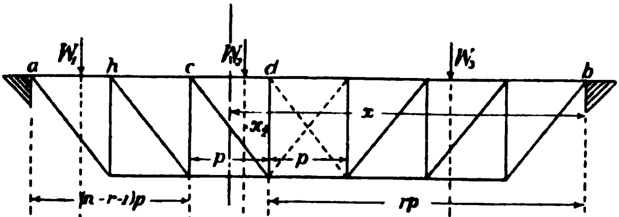

FIG. 117.

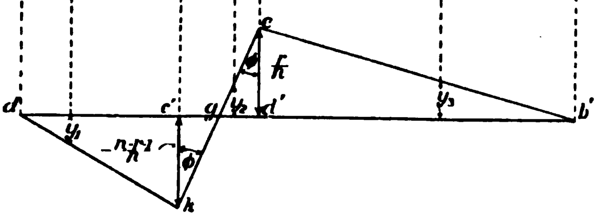

FIG. 118.

Let n denote the number of panels in the truss, and r the number to the right of the point d. The shear in the panel cd for a load moving from b to d will be represented by the influence line eb', where

---

[1] For Cooper's E_{40}, and other standard loading, see Chapter XVI.

$ed' = \frac{r}{n}$. If the load moves from c to a, the influence line is $a'k$, where $c'k = -\frac{n-r-1}{n}$. The shear is negative. The complete influence line is $a'keb'$ (Fig. 118). For a maximum positive shear, for a single concentrated load, it should be placed at d. A uniform load should extend from b to a point over g, and the maximum positive shear will be—

$$w \times \text{area } geb'$$

$c'g = \frac{n-r-1}{n} \tan\phi, gd' = \frac{r}{n}\tan\phi \therefore p = \frac{n-1}{n}\tan\phi$, and $\tan\phi = \frac{np}{n-1}$, $gd' = \frac{r}{n}\left(\frac{np}{n-1}\right) = \frac{rp}{n-1}$. The area $geb' = \frac{r}{2n}\left(rp + \frac{rp}{n-1}\right) = \frac{pr^2}{2n}\left(\frac{n}{n-1}\right)$ $= \frac{pr^2}{2(n-1)}$. The shear $= \frac{wp}{2}\left(\frac{r^2}{n-1}\right)$.

For concentrated loads, let W_1 represent the load in the panel ah, W_2 that in the panel cd, W_3 any load to the right of the point d, and W the total load on the truss. Also, let x denote the distance of the centre of gravity of the loads on the truss from the point b, and x_2 the distance of W_2 from d. The shear at d is—

$$S = \frac{Wx}{l} - \frac{W_2x_2}{p} - W_1.$$

For the position producing the maximum shear at d, $\frac{dS}{dx} = 0$,

$$\text{since } \frac{dS}{dx} = \frac{W}{l} - \frac{W_2}{p} = 0$$

$$\therefore \frac{Wp}{l} = W_2 \text{ and } W = nW_2.$$

To produce the maximum shear there should be no loads in front of c, and the sum of the loads in the panel cd, not including any load which may be over the point d, multiplied by the number of panels in the truss, must be equal—as nearly as possible—to the total load on the truss. If P denotes the sum of the loads in the panel, then—

$$W - nP = lc \text{ minimum}.$$

This criterion may be used to determine the position of a series of concentrated loads, such as the wheels of a locomotive, in order to produce the maximum shear at any point of a truss or girder.

We may use influence lines in connection with the two cases illustrated in Figs. 119 to 121, in which a beam is supported at two points with an overhang at each end. The influence line for the bending moment at a distance μl from the support for a load P equal to unity $\mu(1 - \mu)l$, is shown in Fig. 120. Outside the supports on the overhanging portions, let kl = the distance from the support, The bending moment at P for the loads on the right overhang is—

$$-\ kl(\mu).$$

The bending moment at P for the loads on the left overhang is—

$$-\ kl(1-\mu).$$

The lines are shown in Fig. 120.

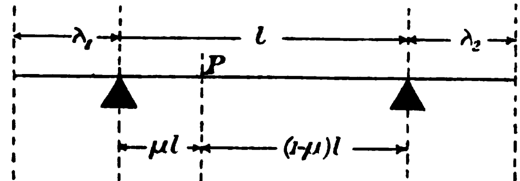

FIG. 119.

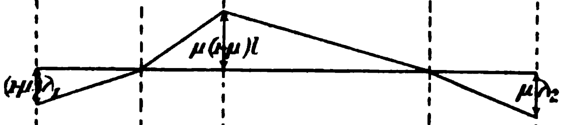

FIG. 120.

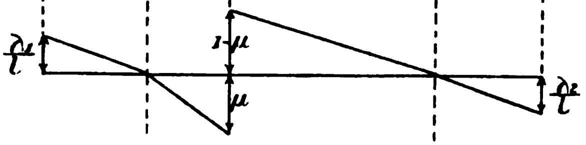

FIG. 121.

For the shear at P the influence line is shown in Fig. 121. For a load outside the supports the shear at P $= k$.

CHAPTER V.

MOMENT OF RESISTANCE—INTENSITY OF HORIZONTAL AND VERTICAL SHEARING STRESSES.

We have seen that when a beam is bent as in Fig. 122, the neutral layer is unchanged in length, and it may be proved that the layers on either side of it are lengthened or shortened in proportion to the distance from the neutral layer. It is assumed that the axis of the beam when unloaded is horizontal, and the loads producing bending act at right angles to the beam. The effect of the loading is to cause the beam to assume a curved form, and sections of the beam which were parallel in the unloaded beam converge towards the centre of curvature in the loaded beam.

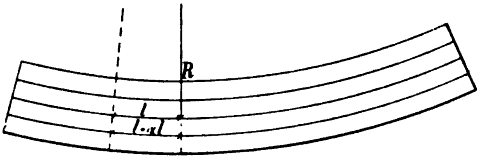

FIG. 122.

Let R = the radius of curvature in the loaded beam.

a = the elongation per unit of length of the fibre situated at a distance y_1 from the neutral layer.

l = the distance between two normal sections of the unloaded beam assumed to be indefinitely near to each other.

Then the fibres at a distance of y from the neutral axis will be extended by an amount al, and the length between two normal sections will be $l + al$, and by similar triangles we have—

$$\frac{R + y_1}{R} = \frac{l + al}{l} \quad \therefore 1 + \frac{y_1}{R} = 1 + a \quad \therefore \frac{y_1}{R} = a$$

and if y_2 denote the distance from the neutral layer to another layer of fibres, and a_2 the extension or strain per unit of length, then—

$$\frac{y_2}{R} = a_2$$

and therefore—

$$\frac{y_1}{y_2} = \frac{a_1}{a_2}$$

i.e., the strain is proportional to the distance from the neutral axis.

We may represent this law graphically as in Fig. 123, where *ac* equals the strain at the extreme fibres, *ab* the depth of the beam, and *a'c'* the strain at the layer of fibres situated at a distance y_1 from the neutral layer or axis.[1] Hence it follows that the resistance which a layer of material will offer to the stresses developed in it will be proportional to the distance of the layer from the neutral axis; also that the neutral axis corresponds with the centre of gravity of the section, provided that the elastic limit of the material is not exceeded.

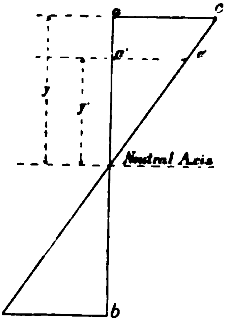

FIG. 123.

If f denote the intensity of stress at a distance y from the neutral axis, and f_1 denote the intensity of stress at a distance y_1 from the neutral axis, then—

$$f_1 : f :: y_1 : y, \text{ and } f_1 = \frac{fy_1}{y}.$$

The moment of resistance of a layer is equal to its direct resistance multiplied by its distance from the neutral axis; hence if b equal the breadth of the layer at a distance y_1 from the neutral axis, and dy its thickness, and f_1 its greatest resistance to stress within the elastic limit of the material, then the moment of resistance of the layer is—

$$bdy \times y_1 \times f_1 = bf_1y_1dy = \frac{bfy_1^2dy}{y},$$

and, since this will be true of every section, the moment of resistance of the whole section is, if b is constant—

$$\frac{b}{y}f\Sigma y_1^2dy, \text{ or } \frac{b}{y}f\int_{-y}^{y} y_1^2dy$$

$$\text{but } b\Sigma y_1^2dy \text{ or } b\int_{-y}^{y} y^2dy = \text{I}$$

where I denotes the moment of inertia of the section with reference to the neutral axis. Therefore the moment of resistance is equal to—

$$\frac{f\text{I}}{y}.$$

This formula is true of any section of any material provided the value of f does not exceed the elastic limit of the material. If the beam is loaded till it is at the breaking-point, the formula is no longer true, as the strains produced are not proportional to the stresses producing them, and the neutral axis does not coincide with the centre of gravity of the section.

If M denote the bending moment which produces fracture, then the equation between the bending moment and moment of resistance is generally written thus—

---

[1] A plane section of the beam before bending becomes a plane section after bending within the limit of proportionality.

$$M = \frac{fI}{y} \qquad (1)$$

from which is obtained—

$$f = \frac{My}{I}. \qquad (2)$$

This is an empirical formula, and f is termed the modulus of rupture, or coefficient of transverse strength. The value of f is not the tensile strength of the material, as may easily be proved by breaking rectangular beams of known dimensions in the testing-machine, and calculating f from equation (2).

In the case of wrought-iron, cast-iron, and steel, the value of f thus obtained will be greater than the tensile strength of the material. Sir B. Baker proved that in the case of steel and wrought-iron beams of rectangular section, the value of f is 70 per cent. in excess of the tensile strength of the material.

If an ordinary rolled iron or steel girder be broken in a similar manner, the value of f will be smaller than in a solid rectangular section, but still greater than the tensile strength of the material. In a well-proportioned wrought-iron or steel-built girder this difference between the value of the modulus of rupture and the tensile resistance of the material is so small that it is generally neglected in calculating the moment of resistance of the section, which is then taken as the moment of resistance of the flanges, the web not being considered with reference to resisting bending. The web is designed to resist the shearing stresses only.

The reason why the modulus of rupture exceeds the tensile strength of the material is probably due to the plastic flow which occurs after the elastic limit has been passed. The extreme fibres in stretching are partially relieved of stress, which is transferred to the layers of fibres nearer to the neutral axis of the section.[1] The neutral axis therefore cannot coincide with the centre of gravity of the section under these circumstances, but must move farther away from the most strained fibres until the point of rupture is reached.

Sir B. Baker has shown, by the results of a large number of experiments on steel rails, that the tensile strength may be deduced from the modulus of rupture thus—

Let A = the area of the section tested, such as a rolled girder, rail, T or bulb iron.
A_1 = the area of a rectangle of the same depth as the section tested, but of a breadth equal to the widest flange.
t = the tensile strength of the material, wrought-iron or steel.

---

[1] A plane section before bending is not a plane section after the limit of proportionality has been exceeded.

Then in the case of solid, rectangular or square sections—

$$f = t\left(1 + \frac{70}{100}\right) = \text{modulus of rupture.}$$

For other sections—

$$f = t\left(1 + \frac{70}{100} \times \frac{A}{A_1}\right).$$

If, however, the elastic limit of the material is not exceeded, the equation $f = \frac{My}{I}$ is a rational formula, and may be used to determine the intensity of stress at the extreme fibre in any section. The value of $\frac{I}{y}$ has been termed the modulus of the section, and may be found when the moment of inertia and the distance y are known.

In order to obtain a true constant of the material from experiments on beams supported at each end and loaded in the centre, we must observe the law of similarity in the form of the test-pieces, or use always a standard test-piece. The International Society for the Unification of Methods of Testing Materials suggests a square bar 1·2 in. side (3 cm.) on supports 39·25 (1 m.) centres. Where it is impossible to have such a bar, the proportion between the length and the side of the square section should be the same, thus : If a denote the side of the square cross-section and l the length, then—

$$\frac{l}{a} = 33{\cdot}3.$$

It should be remembered in cast test-pieces the results will be affected by the presence or absence of the casting skin.[1]

In the case of square or rectangular cross-sections the moment of resistance may be expressed in a more simple manner : Since $I = \frac{1}{12}bd^3$ for a rectangular section about an axis passing through the centre of gravity, and $y = \frac{d}{2}$,

$$\therefore \frac{fI}{y} = \frac{bd^2}{6}f.$$

For beams supported at each end and loaded in the centre—

$$\frac{Wl}{4} = \frac{bd^2}{6}f \therefore f = \frac{3Wl}{2bd^2}.$$

In order that the values of f deduced from experiments on beams should be strictly comparable, it is necessary to preserve the law of similarity in the form and dimensions of the test-pieces.

Moment of Inertia.—The meaning of the term "moment of inertia," as at present used, in relation to a solid body, is as follows : The moment of inertia of a body about a given axis is the limit of the sum of the products of the weight of each of the elementary particles that make up the body by the squares of their distances from a given axis.

---

[1] See tests of cast-iron, Chap. I.

If w_1, w_2, w_3, etc., are the weights of the particles situated at distances r_1, r_2, r_3, etc., respectively, from the axis, the moment of inertia of the body about the given axis is—

$$I = \text{limit of } \Sigma wr^2.$$

The radius of gyration of a body with respect to a given axis is the perpendicular distance from the axis to that point at which, if the whole mass of the body were concentrated, the angular momentum, and hence the moment of inertia, would remain the same as they are in the body itself.

If ρ denote the radius of gyration, the moment of inertia would be, when the mass is concentrated—

$$\rho^2\Sigma w,$$

hence we have—

$$\rho^2\Sigma w = \Sigma wr^2 = I$$

and—

$$\rho^2 = \frac{\Sigma wr^2}{\Sigma w} = \frac{I}{W}$$

where W = the weight of the whole body.

The moment of inertia of plane surfaces about an axis is the limit of the sum of the products of the elementary areas into which the body may be conceived to be divided by the squares of their distances from the axis in question. Similarly, if A denote the area of a plane surface, and ρ the radius of gyration—

$$\rho^2 = \frac{I}{A}.$$

If the plane surface under consideration be referred to a pair of axes *ox* and *oy* in the same plane, the moment of inertia about *ox* will be—

$$\int\int y^2dxdy_1$$

and the moment of inertia about *oy* will be—

$$\int\int x^2dxdy_1.$$

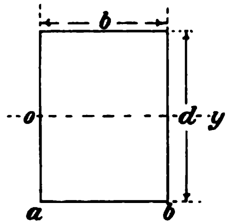

FIG. 124.

Find the moment of inertia of a rectangle about an axis passing through its centre of gravity, *oy* (Fig. 124)—

$$I = \int_{-\frac{d}{2}}^{\frac{d}{2}}\int_{0}^{b} x^2dxdy = b\int_{-\frac{d}{2}}^{\frac{d}{2}} x^2dx = \frac{bd^3}{12}.$$

In a similar manner, if the axis passed through one end, such as *ab*, we have—

$$I = \int_{0}^{d}\int_{0}^{b} x^2dxdy = b\int_{0}^{d} x^2dx = \frac{bd^3}{3}.$$

The moment of inertia may be found from the following graphical construction.[1] Let Fig. 125 denote the cross-section. Divide up the cross-section, as shown, into parallel layers, and determine the area of each. Lay off lines 1, 2, 3 . . . 6 (Fig. 126), equal to these areas to some convenient scale, so that AB represents the total area of the cross-section. Make $BC = \frac{AB}{2}$. Draw horizontal lines through the centres of gravity of each of the small portions of the cross-section, and from any point *o* (Fig. 127) draw lines parallel to the strings *d*, *e*, *f*, *g*, *h*, *j*, forming the polygon in the manner shown. The intersection of *a* with *j* locates the neutral axis. The area of the cross-section may be denoted by A and the area of the polygon by A′; then—

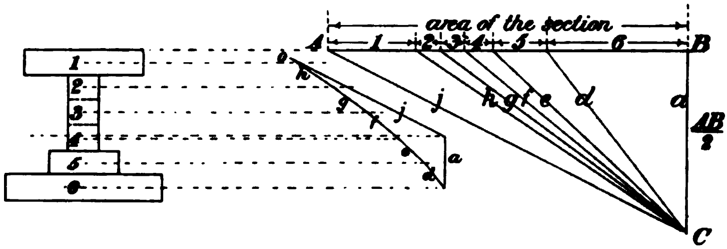

FIG. 125. FIG. 127. FIG. 126.

$$I = AA'$$

$$\rho^2 = \frac{I}{A} = A'.$$

For symmetrical sections the moment of inertia may be found thus :—

The section should be divided up into rectangles and dimensioned as indicated in Fig. 128, and since the moment of inertia of a

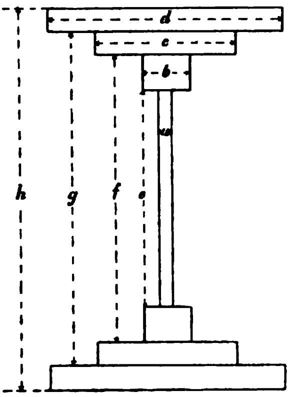

FIG. 128.

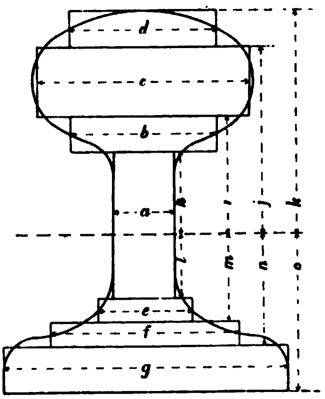

FIG. 129.

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[1] Molesworth.

rectangle about an axis passing through its centre of gravity is $\frac{1}{12}bd^3$, the moment of inertia of the whole section is—

$$I = \tfrac{1}{12}\{ae^3 + b(f^3 - e^3) + c(g^3 - f^3) + d(h^3 - g^3)\}.$$

For unsymmetrical sections the moment of inertia may be found thus :—

The section should be divided up into rectangles and dimensioned as indicated in Fig. 129, and, since the moment of inertia of a rectangle about an axis passing through one end is $\frac{1}{3}bd^3$, the moment of inertia of the whole section is—

$$I = \tfrac{1}{3}\{a(h^3 + l^3) + b(i^3 - h^3) + c(j^3 - i^3) + d(k^3 - j^3) + e(m^3 - l^3) + f(n^3 - m^3) + g(o^3 - n^3)\}.$$

As an example of a symmetrical section, take a rolled iron girder, 12′ × 6″ × ½″, as shown in Fig. 130.

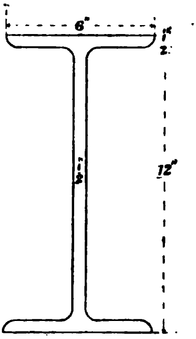

FIG. 130.

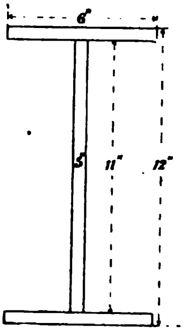

FIG. 131.

The girder should be dimensioned as shown in Fig. 131, then the moment of inertia may be written down as follows :—

$$I = \tfrac{1}{12}\{\cdot5(11)^3 + 6(\overline{12^3 - 11^3})\} = 254 \text{ nearly.}$$

The moment of resistance may be found thus :—

$$\text{Moment of resistance} = \frac{fI}{y} = \frac{254}{6} \times f = 42{\cdot}3f.$$

If f be taken as 5 tons per sq. in.—

$$\text{Moment of resistance} = 42{\cdot}3 \times 5 = 211{\cdot}5 \text{ inch-tons.}$$

If it is desired to know the greatest load it will safely carry uniformly distributed over a span of 12 ft., we proceed as follows : Let W denote the total distributed load. The bending moment is—

$$\frac{Wl}{8} = \frac{W \times 12 \times 12}{8} = 18W \text{ inch-tons}$$

$$\therefore 18W = 211{\cdot}5, \text{ and } W = 11{\cdot}75 \text{ tons.}$$

The maximum shearing stress is $\frac{11\cdot75}{2} = 5\cdot87$ tons; and the intensity of shearing stress is $\frac{5\cdot87}{5\cdot5} = 1$ ton (about) per sq. in.

If we require to know the maximum intensity of stress produced by a central load of 5 tons, we have—

$$\frac{\text{W}l}{4} = 42\cdot3f = \frac{5 \times 144}{4} = 180$$

$$\therefore f = \frac{180}{42\cdot3} = 4\cdot2 \text{ tons per sq. in. (nearly).}$$

The maximum intensity of shearing stress is $\frac{2\cdot5}{5\cdot5} = \cdot45$ tons per sq. in.

Moment of Resistance of Built Girders.—In plate web girders, consisting of horizontal flange plates united to a thin vertical web-plate by means of double angles top and bottom riveted to both the web and the flanges, the moment of resistance of the web-plate is very small compared with that of the flanges, and may be neglected without causing any serious error. In box girders, in which there are two or more web-plates, the webs should be considered by finding the moment of inertia and using the equation—

$$\text{Moment of resistance} = \frac{f\text{I}}{y}.$$

In lattice girders the moment of resistance is usually calculated on the assumption that the flanges resist the bending stresses, while the web resists the shearing stresses as in the case of plate web girders. The moment of resistance in such cases may be expressed as follows:—

Let f = the intensity of working stress, generally taken as the working tensile stress.
a = the area of the tension flange after deducting for rivet-holes.
d = the effective depth of the girder.

Then we have—

$$\text{Moment of resistance} = fad.$$

The bottom flange is designed by means of the above equation, and the top flange is usually made similar to it in every respect.

The effective depth of the girder is generally taken as the distance between the centres of gravity of the top and bottom flanges whenever the flanges of the girder consist of both horizontal plates and angles; but when there are no horizontal plates, and the angles form the flanges, the effective depth is more correctly measured between the centres of the rivets uniting the web to the flanges.

Fig. 132 represents a plate web girder, which will be used as an example of the foregoing method of finding the moment of resistance, thus—

The area of the tension flange, after deducting two rivets 1 in. diameter,[1] is—

Three plates $= 3(12 - 2 \times 1)\frac{1}{2} = 15$ sq. in.
Two angles $= 2(7 - 1)\frac{1}{2} = 6$ sq. in.
$\therefore A = 21$ sq. in.

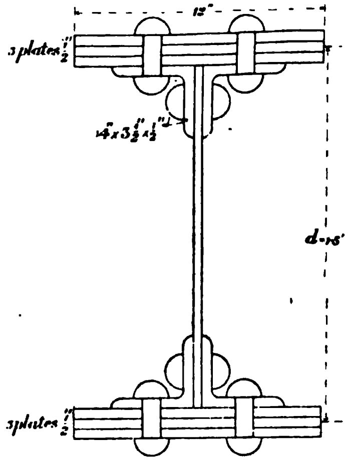

FIG. 132.

The effective depth measured between the centres of gravity of the flanges is 1·5 ft. The working stress may be taken at 5 tons; then—

Moment of resistance $= fad = 5 \times 21 \times 1{\cdot}5 = 157{\cdot}5$ foot-tons.

The safe distributed load on such a girder on an effective span of 16 ft. is found thus—

$$\frac{Wl}{8} = \frac{W \times 16}{8} = 2W = 157{\cdot}5$$

$\therefore W = 78{\cdot}75$ tons.

This weight may be carried without the intensity of stress exceeding 5 tons on the flanges, provided that the girder is properly designed in other respects.

As an example of an unsymmetrical section, take a cast-iron girder of the form and dimensions shown in Fig. 133.

Let A_1 = area of top flange $= 5 \times 1\frac{1}{2} = 7{\cdot}5$ sq. in.
A_2 = ,, bottom flange $= 15 \times 2 = 30{\cdot}0$ sq. in.
A_3 = ,, web $= 17{\cdot}5 \times 1{\cdot}75 = 30{\cdot}625$ sq. in.
$\bar{x}$ = distance of the centre of gravity of the section from the bottom edge of the bottom flange.
x_1 = distance of the centre of gravity of the top flange from the same edge.
x_2 = distance of the centre of the bottom flange from the same edge.
x_3 = distance of the centre of the web from the same edge.

$$\bar{x}(A_1 + A_2 + A_3) = A_1x_1 + A_2x_2 + A_3x_3$$

$$x(68{\cdot}125) = 7{\cdot}5 \times 20{\cdot}25 + 30 \times 1 + 30{\cdot}625 \times 10{\cdot}75$$

$$\therefore \bar{x} = \frac{511{\cdot}094}{68{\cdot}125} = 7{\cdot}5.$$

The section of the girder should be dimensioned as indicated in

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[1] See pages 299 and 351.

Fig. 134, for the purpose of finding the moment of inertia with reference to the axis passing through the centre of gravity, which may be now written down—

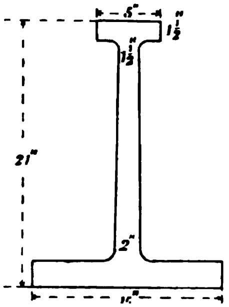

FIG. 133.

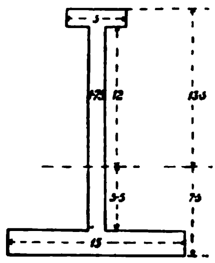

FIG. 134.

$$\begin{aligned} \mathrm{I} &= \tfrac{1}{3}\{1\cdot75(\overline{12^3} + \overline{5\cdot5^3}) + 5(\overline{13\cdot5^3} - \overline{12^3}) + 15(\overline{7\cdot5^3} - \overline{5\cdot5^3})\} \\ &= \tfrac{1}{3}(3315\cdot156 + 3161\cdot875 + 3832\cdot5) \\ &= 3436\cdot531. \end{aligned}$$

In this case we have the moment of resistance for the tension flange.

$$\text{Moment of resistance} = \frac{f_t\mathrm{I}}{y_t} = \frac{f_t \times 3436\cdot531}{7\cdot5}.$$

The moment of resistance of the compressive flange is—

$$\text{Moment of resistance} = \frac{f_c\mathrm{I}}{y_c} = \frac{f_c \times 3436\cdot531}{13\cdot5}.$$

If we allow 2½ tons per sq. in. as the maximum intensity of safe tensile stress on the extreme fibre, we have—

$$\text{Moment of resistance} = \frac{2\cdot5 \times 3436\cdot531}{7\cdot5} = 1145\cdot5 \text{ inch-tons.}$$

If we allow 7·5 tons per sq. in. as the maximum intensity of compressive stress, we have—

$$\text{Moment of resistance} = \frac{7\cdot5 \times 3436\cdot531}{13\cdot5} = 1909 \text{ inch-tons.}$$

Hence since the moment of resistance of the tension flange is smaller than that of the compression flange, we must equate the former with the bending moment. The strength of cast-iron beams is, however, usually calculated by the following formula, which neglects the area of the web; thus—

Let f = the mean intensity of stress on the bottom or tension flange.

a = the area of the tension flange.

d = the depth measured between the centres of gravity of the flanges.

Then—

Moment of resistance $= fad = 30 \times 19{\cdot}25f = 577{\cdot}5f$.

Referring to Fig. 134, it will be seen that if 2·5 tons is the maximum intensity of tensile stress at the extreme fibre, the intensity of stress at the inner edge of the bottom flange is—

$$\frac{2{\cdot}5 \times 5{\cdot}5}{7{\cdot}5} = 1{\cdot}83 \text{ tons per sq. in.}$$

The mean intensity of stress is therefore—

$$\frac{2{\cdot}5 + 1{\cdot}83}{2} = 2{\cdot}16 \text{ tons per sq. in.}$$

Hence moment of resistance $= 577{\cdot}5 \times 2{\cdot}16 = 1247{\cdot}4$ inch-tons, which is slightly greater than before.

The ultimate resistance of cast-iron such as would be used in girder work is as follows :—

Tension	9	tons per sq. in.
Compression	43	,, ,,
Shearing	8½	,, ,,

It follows, therefore, that in cast-iron beams of similar proportions to the one shown in Fig. 133, the tension flange will always be the weaker of the two. If the area of the top flange is diminished, so that the moment of resistance in compression is more nearly equal to that of the tension flange, the form of the section will not be so suitable for casting, as the thinner top flange will cool first, and there will be a danger of a flaw in consequence of the subsequent contraction of the thick bottom flange. For the same reason the web is made tapering, and of a thickness equal to that of the flange it joins. Again, the flanges are tapered so as to draw easily from the sand. The thickness of the web is always greatly in excess of that necessary to resist the shearing stresses, and feathers or stiffeners should be avoided. The top flange, if reduced to the size required by theory, would probably buckle laterally.

If the span of the girder, measured between the centre of its bearings, is 21 feet 6 inches, what is the greatest safe load it will carry uniformly distributed over its length?

Here—

$$\frac{Wl}{8} = \frac{W \times 21{\cdot}5 \times 12}{8} = 32{\cdot}25W = \text{maximum bending moment.}$$

If moment of resistance = 1146 inch-tons

$$32{\cdot}25\text{W} = 1146$$

$$\therefore \text{W} = \frac{1146}{32{\cdot}25} = 35{\cdot}53 \text{ tons.}$$

The greatest shearing stress will occur over the supports, and is—

$$\frac{35{\cdot}53}{2} = 17{\cdot}76 \text{ tons.}$$

If the beam is of uniform depth and the flanges of uniform width, the maximum intensity of shearing stress is—

$$\frac{17{\cdot}76}{\text{area of web}} = \frac{17{\cdot}76}{30{\cdot}625} = {\cdot}58 \text{ tons per sq. in.}$$

The beam may be varied in depth by curving the upper flange, or the width of the flanges may be varied, or both. If the safe intensity of working stress in shear is 1 ton per sq. in., the depth over the supports may be reduced thus: where d denotes the reduced depth, and $1{\cdot}75d$ the area of the web—

$$\frac{17{\cdot}76}{1{\cdot}75d} = 1$$

$$\therefore d = 10 \text{ in.}$$

Modulus of a Section.—The modulus of the section of any material depends upon its form and dimensions. In symmetrical sections the relative resistance to bending per unit of area, in regard to a plane perpendicular to the axis about which the moment of inertia has been calculated (the axis passing through the centre of gravity of the section), may be found by dividing the modulus of the section by the area of the section. In unsymmetrical sections the same applies, if we remember that the distance from the most strained fibre must be that which will give the smaller moment of resistance.

We have seen that the resistance of a layer of fibres is directly proportional to the area of the layer and to the distance from the neutral axis. If the resistance of the fibres at the extreme (or most strained) layer be denoted by unity, the resistance of the other layers may be represented graphically by drawing lines across the section, the lengths of which are reduced in the ratio of their distances from the neutral axis. In the case of the rectangle, Fig. 135, the lines are enclosed by two triangles, which represent the equivalent figures of uniform stress on the assumption that the stress is constant and the section varying, and the modulus of the section is equal to the area of one of the triangles multiplied by the distance between the centres of gravity of the triangles.

If a denote the area of one of the triangles, and d' the distance between the centres of gravity, we have the modulus of the section, M, thus—

$$M = ad'$$

$$\text{Area of triangle} = \frac{bd}{4}\,;\ \text{and}\ d' = \frac{2d}{3}$$

$$\therefore\ M = ad' = \frac{bd^2}{6},\ \text{as before.}$$

If the same method be applied to a circular section, the figure of uniform stress will be as shown in Fig. 136. The points through which these curves are drawn are found thus: Lines are drawn across the circle, such as *ab*, and perpendiculars let fall from their extremities, such as *bd* and *ac*, on to a horizontal line drawn through the top of the section; *c* and *d* are then joined to *o*, intersecting *ab* in *a'* and *b'*; *a'* and *b'* are points in the curve of uniform stress. In a similar manner other points in the curves may be determined. Figs. 137 to 145 show the same method applied to other symmetrical sections.

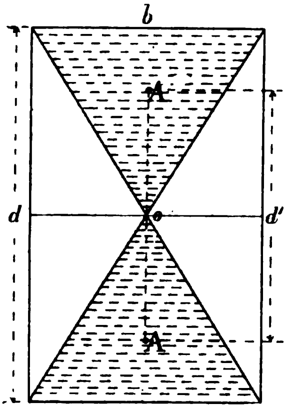

FIG. 135.

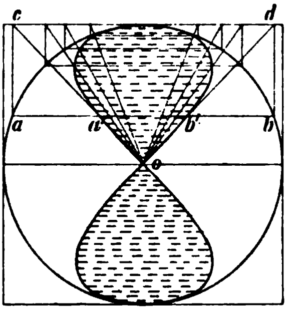

FIG. 136.

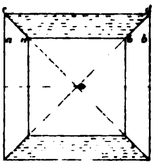

FIG. 137.

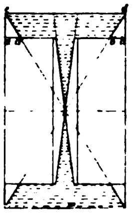
FIG. 138.

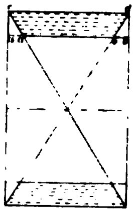
FIG. 139.

In the case of unsymmetrical sections, such as a rail, Fig. 146, proceed as follows: Find the centre of gravity of the section, *o*; draw a line *ab* parallel to the line *cd*, and at a distance from *o* equal to the distance *ef* from *o*. Since *cd* is nearer to *o* than *ef*, it will offer a

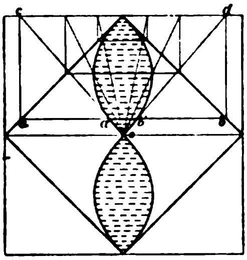

FIG. 140.

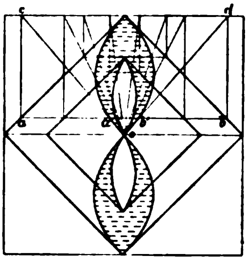

FIG. 141.

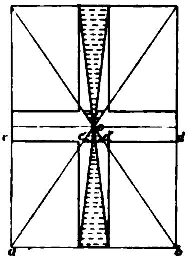

FIG. 142.

correspondingly smaller resistance, which is found by joining *a* and *b* to *o*, and noting the intersections *c′* and *d′* with *cd*. In a similar

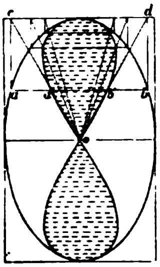

FIG. 143.

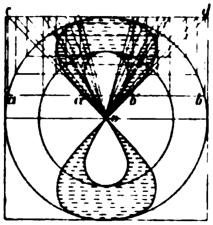

FIG. 144.

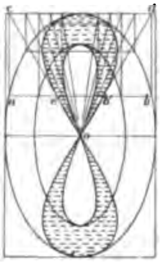

FIG. 145.

manner other points in the figure are determined. Figs. 147 to 150 show other unsymmetrical sections.

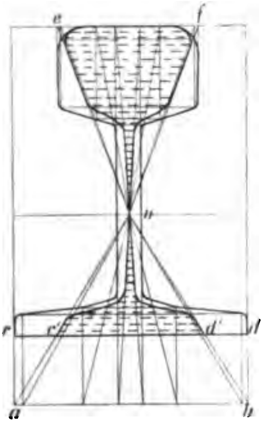

FIG. 146.

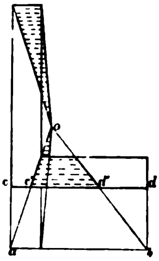

FIG. 147.

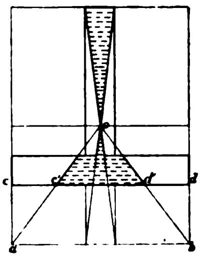

FIG. 148.

Sir Benjamin Baker suggested[1] that the areas of the shaded

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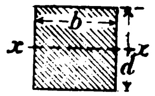
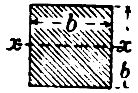
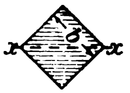
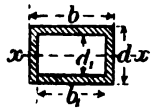
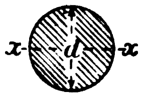
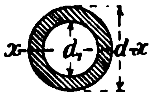
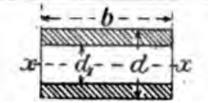
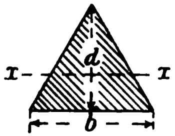
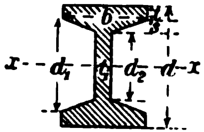
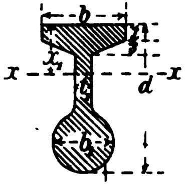
[1] *Proc. Inst. C. E.*, vol. lxii., p. 251.

TABLE XXX.

VALUES OF I AND y, FOR MOMENT OF RESISTANCE $= \frac{fI}{y}$.

Form of the section.	Area of the section.	Distance of the centre of gravity of the section from the extreme fibre = y.	Moment of inertia about the axis xx passing through the centre of gravity.	Moment of resistance of the section $= \frac{fI}{y}$
	bd	$\frac{d}{2}$	$\frac{bd^3}{12}$	$\frac{bd^2f}{6}$
	b^2	$\frac{b}{2}$	$\frac{b^4}{12}$	$\frac{b^3f}{6}$
	b^2	$\frac{b\sqrt{2}}{2}$	$\frac{b^4}{12}$	$\frac{b^3f}{6\sqrt{2}}$
	$bd - b_1d_1$	$\frac{d}{2}$	$\frac{bd^3 - b_1d_1^3}{12}$	$(bd^3 - b_1d_1^3)\frac{f}{6d}$
	$\frac{\pi d^2}{4}$	$\frac{d}{2}$	$\frac{\pi d^4}{64}$	$\frac{\pi}{32}d^3f$

	$\frac{\pi}{4}(d^2 - d_1^2)$	$\frac{d}{2}$	$\frac{\pi}{64}(d^4 - d_1^4)$	$\frac{\pi}{32}(d^4 - d_1^4)\frac{f}{d}$
	$b(d - d_1)$	$\frac{d}{2}$	$\frac{b}{12}(d^3 - d_1^3)$	$\frac{b}{6d}(d^3 - d_1^3)f$
	$\frac{bd}{2}$	$\frac{2d}{3}$	$\frac{bd^3}{36}$	$\frac{bd^2f}{24}$
 b s x d₁ t₁ d₂ d x	$2bt + d_1t_1 + s(b - t_1)$ $s = \frac{b - t_1}{2}r$ $s = \frac{br}{2}$ approx.	$\frac{d}{2}$	$\frac{bd^3 - \frac{1}{4r}(d_1^4 - d_2^4)}{12}$	$\frac{fI}{y}$
 b s x x₁ t₁ d x b₁ t	$\frac{\pi}{4}b_1^2 + (d - b_1)t_1 + (b - t_1)\left(t + \frac{s}{2}\right)$	$\frac{\frac{\pi}{4}b_1^2(2d - b_1) + t_1(d - b_1)^2}{2A}$ $+ \frac{(b - t_1)(t^2 + st + \frac{1}{3}s^2)}{2A}$	$\frac{\pi}{4}b_1^2\left\{\frac{b_1^2}{16} + \left(d - \frac{b_1}{2}\right)^2\right\}$ $+ \frac{t_1(d - b_1)^3}{3}$ $+ \frac{(t + s)^4 - t^4}{6r}$ $- Ax_1^2$	$\frac{fI}{y}$

portions may be obtained by cutting them out from a sheet of some material of uniform thickness and weighing them against strips of the same material 1 in. wide; thus, if zinc is the material, and a strip 10 in. long by 1 in. wide is used, pieces may be cut off and placed in the scale until an exact balance is obtained; the remainder subtracted from 10 in. gives the required area in sq. in. The centres of gravity may be obtained by balancing each shaded area on a knife-edge. The accuracy of the work is proved by the equality of the two shaded areas on each side of the neutral axis, as demonstrated by weighing one against the other. Table XXX. gives the moments of resistance for various cross-sections.

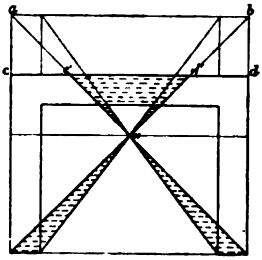


Fig. 149.

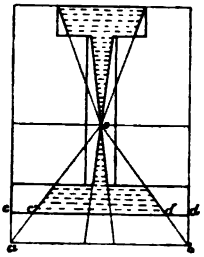


Fig. 150.

The following tables give the dimensions, weights per ft., and moduli of section of some steel girders made by a few well-known manufacturers. The central breaking weight for any span may be found by the following equation as given in the beginning of the chapter:—

$$\frac{fI}{y} = \frac{Wl}{4}$$

or, for a distributed load—

$$\frac{fI}{y} = \frac{Wl}{8}$$

where W includes the weight of the girder, l being the span in in.

The value of f may be taken at 28 tons for steel.

The strength of the steel girders manufactured by Messrs. Dormon, Long & Co., as given in their circulars, are calculated with 32 tons as the value of f.

Table XXXI.

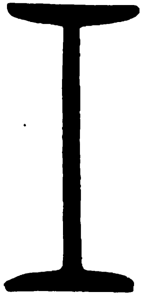
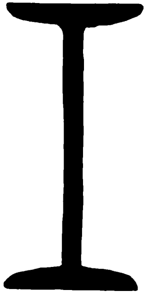
Description.	Dimensions in inches.				Sectional area in square inches.	Weight per lineal foot in pounds.	Modulus of section $= \frac{I}{y}$
	Depth of girder.	Width of flanges.	Average thickness of flanges.	Minimum thickness of web rolled.			
 Rolled steel girders manufactured by Carnegie, Phipps & Co., Lim., Pittsburg, Pa.	20·0	7·00	0·900	0·60	23·5	80·0	144·9
	20·0	6·25	0·765	0·50	18·8	64·0	114·6
	15·0	6·31	1·035	0·67	22·1	75·0	101·0
	15·0	6·04	0·900	0·54	17·6	60·0	85·9
	15·0	5·75	0·750	0·45	14·7	50·0	70·6
	15·0	5·50	0·590	0·40	12·0	41·0	56·6
	12·0	5·50	0·690	0·39	11·7	40·0	46·9
	12·0	5·25	0·535	0·35	9·4	32·0	37·0
	10·0	5·00	0·645	0·37	9·7	33·0	32·3
	10·0	4·75	0·485	0·32	7·5	25·5	24·7
	9·0	4·75	0·585	0·31	7·9	27·0	24·6
	9·0	4·50	0·440	0·27	6·2	21·0	18·7
	8·0	4·50	0·510	0·27	6·5	22·0	18·0
	8·0	4·25	0·410	0·25	5·3	18·0	14·4
	7·0	4·25	0·500	0·27	5·9	20·0	14·2
	7·0	4·00	0·390	0·23	4·6	15·5	11·0
	6·0	3·625	0·465	0·26	4·7	16·0	9·54
	6·0	3·50	0·375	0·23	3·8	13·0	7·83
	5·0	3·13	0·435	0·26	3·8	13·0	6·28
	5·0	3·00	0·335	0·22	3·0	10·0	4·96
	4·0	2·75	0·395	0·24	2·9	10·0	3·86
	4·0	2·625	0·290	0·20	2·2	7·5	2·95

TABLE XXXIA.

Description.	Dimensions in inches.				Sectional area in square inches.	Weight per lineal foot in pounds.	Modulus of section $= \frac{I}{y}$
	Depth of girder.	Width of flanges.	Average thickness of flanges.	Minimum thickness of web rolled.			
 Rolled steel girders manufactured by Messrs. Dorman, Long & Co.	20·0	8·26	0·97	0·76	29·67	100·0	182·5
	20·0	8·18	0·97	0·68	28·24	95·0	177·2
	20·0	8·11	0·97	0·61	26·78	90·0	172·5
	18·0	6·90	0·94	0·61	22·88	78·0	128·6
	16·0	5·95	0·82	0·54	17·51	59·0	87·0
	15·0	6·01	0·81	0·64	16·96	57·0	82·5
	15·0	5·06	0·80	0·50	14·88	50·0	67·6
	14·0	5·87	0·81	0·50	15·76	53·0	70·4
	12·0	5·99	0·87	0·49	15·47	52·0	61·3
	12·0	4·98	0·65	0·48	11·60	39·0	43·0
	10·0	6·01	0·70	0·51	12·76	43·0	41·9
	10·0	5·03	0·66	0·52	10·26	34·5	34·7
	10·0	4·38	0·66	0·38	8·63	29·0	29·4
	9·0	6·90	0·81	0·66	10·06	54·0	46·7
	9·0	3·63	0·50	0·38	6·70	22·5	18·37
	8·0	5·90	0·62	0·40	10·12	34·0	27·54
	8·0	4·91	0·62	0·35	8·48	28·5	23·02
	8·0	3·91	0·56	0·31	6·55	22·0	18·91
	7·0	3·70	0·46	0·32	5·36	18·0	12·10
	6·25	3·38	0·50	0·26	4·76	16·0	9·96
	6·0	4·94	0·50	0·44	7·14	24·0	14·01
	6·0	2·99	0·50	0·29	4·46	15·0	8·60
	6·0	1·99	0·38	0·31	3·12	10·5	5·23
	5·5	1·99	0·38	0·31	2·97	10·0	4·61
	5·0	4·90	0·5625	0·33	6·82	23·0	11·55
	5·0	4·35	0·58	0·35	6·39	21·5	9·91
	5·0	3·04	0·46	0·30	4·02	13·5	5·78
	4·75	1·72	0·35	0·32	2·53	8·5	2·46
	4·62	3·04	0·40	0·29	3·57	12·0	4·70
	4·0	3·04	0·407	0·30	3·42	11·5	4·01
	4·0	1·76	0·35	0·26	2·08	9·0	2·06
	3·5	2·84	0·30	0·15	2·68	9·0	2·50
	3·5	1·51	0·30	0·20	1·49	5·0	1·33
	3·0	2·81	0·40	0·13	2·53	8·5	2·55
	3·0	1·30	0·25	0·21	1·19	4·0	0·82

Distribution of Shearing Stress in the Cross-section of a Beam.—Let us consider the distribution of shearing stress on a solid beam of rectangular section, supported at both ends and loaded (Figs. 151 and 152).

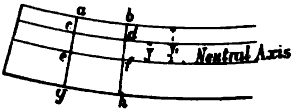


FIG. 151.

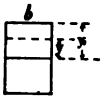


FIG. 152.

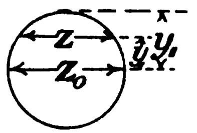


FIG. 153.

Let M = the bending moment of section *aceg*.

M_1 = ,, ,, ,, *bdfh*.

m = the distance between the sections *aceg* and *bdfh*.

Generally M_1 will be greater than M.

Let y_1 = the distance of the top layer *ab* from the neutral axis, and *y* the distance *cd*.

b = the breadth of the beam.

The mean intensity of stress at *a*, due to bending moment M, is $\frac{M}{I}y_1$; at *c*, $\frac{M}{I}y$; at *b*, $\frac{M_1}{I}y_1$; at *d*, $\frac{M_1}{I}y$. The mean intensity of stress on *bd* is $\frac{M_1}{2I}(y_1 + y)$; and on *ac* is $\frac{M}{2I}(y_1 + y)$.

The total force with which the portion *bd* is pushed towards *ac* is equal to the difference between the mean intensities of stress multiplied by the area over which it is distributed, *viz.* :—

$$\frac{M_1 - M}{2I}(y_1 + y) \times (y_1 - y)b = \frac{M_1 - M}{2I}(y_1^2 - y^2)b.$$

The intensity of shearing stress along *cd* is equal to this force divided by the area of the portion *cd*, *i.e.*, *mb*. Therefore the intensity of shearing stress along *cd* is—

$$\frac{M_1 - M}{2Ibm}(y_1^2 - y^2)b = \frac{M_1 - M}{2Im}(y_1^2 - y^2).$$

But if S denote the total shearing stress on the section *bdfh*, it may be shown that—

$$\frac{M_1 - M}{m} = \text{S where } m \text{ is indefinitely small.}$$

$$\therefore \text{Intensity of shearing stress} = \frac{S}{2I}(y_1^2 - y^2).$$

The maximum intensity of stress occurs along the neutral axis, where $y = o$, and is—

$$\frac{Sy_1^2}{2I}.$$

In rectangular beams $y_1 = \frac{d}{2}$, and $I = \frac{bd^3}{12}$, hence the maximum intensity of shearing stress is—

$$\frac{3S}{2bd}.$$

Fig. 154 represents the distribution of direct and shearing stresses on a beam of rectangular section.

The shaded area, *abcd*, is a graphical representation of the intensity of shearing stress at a distance y from the neutral axis; it equals—

$$\frac{S}{2I}(y_1^2 - y^2) = \frac{6S}{bd^3}\left(\frac{d^2}{4} - y^2\right).$$

$$\text{If } y = \frac{d}{3}, \text{ we have } \frac{6S}{bd^3}\left(\frac{d^2}{4} - \frac{d^2}{9}\right) = \frac{5S}{6bd}.$$

$$\text{If } y = \frac{d}{6}, \text{ we have } \frac{6S}{bd^3}\left(\frac{d^2}{4} - \frac{d^2}{36}\right) = \frac{4S}{3bd}.$$

The curve representing the distribution of shearing stress is a parabola, the ordinates of which are shown in Fig. 154.

The foregoing investigation shows that the intensity of horizontal shearing stress is greatest along the plane of the neutral axis, and, from the equality of the intensities of shearing stresses on planes at right angles to each other, it follows that the intensity of vertical shearing stress must equal the intensity of horizontal shearing stress at every point in the beam. In timber beams the resistance to shearing along the fibres is so small, especially in spruce and pine timbers, that it is necessary to so proportion the section that resistance to shearing along the neutral axis is at least as great as the resistance to bending. This can always be done when the modulus of rupture and shearing resistance along the grain are known.

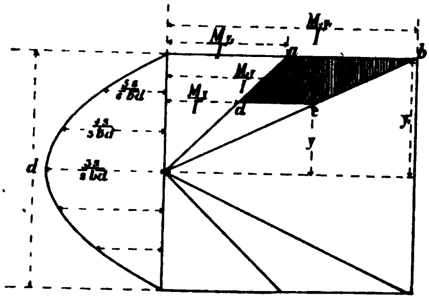


FIG. 154.

Breadth of the Beam Variable.—The following investigation applies to any shape of the cross-section of a beam. Let z denote the breadth of the cross-section at a distance y from the neutral surface, f the intensity of direct stress, and σ, the intensity of shearing stress. If the distance between the two planes *bh* and *ag* (Fig. 151) be denoted by dx, where dx is indefinitely small, then—

$$f = \frac{My}{I}.$$

The amount of direct stress in the sectional plane between *b* and *d* (Fig. 151) is—

$$\frac{\mathrm{M}}{\mathrm{I}}\int_{y}^{y_1} yzdy.$$

The horizontal force pressing the solid *abcd* towards *ac* is the excess of the value of the above quantity for *bd* over its value for *ac*, since the bending moment at *bh* is greater than the bending moment at *ag*, therefore—

$$\frac{\mathrm{M}_1 - \mathrm{M}}{dx} = \mathrm{S}.$$

The horizontal force in equilibrium with the shearing force S is—

$$\frac{\mathrm{M}_1 - \mathrm{M}}{\mathrm{I}}\int_{y}^{y_1} yzdy = \frac{\mathrm{S}dx}{\mathrm{I}}\int_{y}^{y_1} yzdy.$$

Dividing this quantity by the area of the plane *cd*, or *zdx*, the intensity of shearing stress is—

$$\sigma_s = \frac{\mathrm{S}}{\mathrm{I}z}\int_{y}^{y_1} yzdy.$$

The maximum value at the neutral layer is—

$$\sigma_{s\,\text{max.}} = \frac{\mathrm{S}}{\mathrm{I}z_o}\int_{o}^{y_1} yzdy$$

where z_o denotes the breadth at the neutral layer. Fig. 153.

The same results are obtained whether the upper or the lower surface of the beam be taken as the limit of integration indicated by y_1, the complete integral, $\int yzdy$, for the whole section being equal to zero, because *y* is measured from the neutral surface which traverses the centre of gravity of the section. In a rectangular cross-section, $z = z_o = b$, and $y_1 = \frac{d}{2}$, also $\mathrm{I} = \frac{bd^3}{12}$, so that the maximum intensity of shearing stress at the neutral layer is—

$$\sigma_{s\,\text{max.}} = \frac{\mathrm{S}}{\mathrm{I}}\int_{o}^{\frac{d}{2}} ydy = \frac{3\mathrm{S}}{2bd},\ \text{as already found.}$$

In the case of a circle or an ellipse it can be shown that the ratio of the mean intensity of shearing stress at the neutral layer, to the mean intensity of shearing stress in the cross-section is equal to $\frac{4}{3}$. In the case of a hollow rectangle and rolled girder sections, the ratio is—

$$\frac{3}{2}\,\frac{(bd - b_1d_1)(bd^2 - b_1d_1^2)}{(b - b_1)(bd^3 - b_1d_1^3)}$$

where the area of the cross-section is $bd - b_1d_1$, and the value of $z_o = b - b_1$.

For a hollow square the ratio is—

$$\tfrac{3}{2}\left\{1 + \frac{bb_1}{b^2 + b_1^2}\right\}$$

where the area of the cross-section is $b^2 - b_1^2$ and $z_o = b - b_1$.

For a hollow circle of diameters d and d_1 the ratio is—

$$\tfrac{4}{3}\left\{1 + \frac{dd_1}{d^2 + d_1^2}\right\}.$$

If W denote the total load uniformly distributed over the length of the beam; l the span, b the breadth, and d the depth of the beam; f the modulus of rupture; S_h the resistance to shearing along the grain; then—

$$W = \frac{4bd^2f}{3l}, \text{ and } S_h = \frac{3S}{2bd} = \frac{3W}{4bd}$$

$$\therefore W = \frac{4bdS_h}{3}$$

$$\therefore \frac{4bd^2f}{3l} = \frac{4bdS_h}{3}$$

$$\text{and } \frac{d}{l} = \frac{S_h}{f}.$$

If the load is concentrated at the centre, as is usually the case when beams are tested—

$$\frac{d}{l} = \frac{2S_h}{f}.$$

The values of S and f vary with the size of the specimen tested, being always smaller for large specimens than for small.

Professor Lanza has shown, by experimenting with spruce and yellow pine timber of large scantlings, such as are used in ordinary building construction, that the average values of f are—

For spruce	4451 lbs. per sq. in.
For yellow pine	7486 „ „

The values of S_h in a large number of specimens, which failed by shearing along the grain, were—

For spruce	191 lbs. per sq. in.
For yellow pine	248 „ „

These values for S_h are less than those obtained by direct experiments on shearing resistance, as in these cases the specimen is compelled to shear at a particular plane, whereas in testing an ordinary beam failure naturally occurs along the plane of least resistance. In the experiments referred to, failure occurred as often by shearing along the grain as by direct tensile stress developed at the extreme fibre. Using the average values given, we have for spruce—

$$\frac{d}{l} = \frac{191}{4451} = \frac{1}{23} \text{ for a distributed load}$$

$$\text{and } \frac{1}{11 \cdot 5} \text{ for a central load.}$$

For yellow pine—

$$\frac{d}{l} = \frac{248}{7486} = \frac{1}{30} \text{ for a distributed load}$$

$$\text{and } \frac{1}{15} \text{ for a central load.}$$

These experiments are confirmed by those made at the Watertown Arsenal and by Prof. Bauschinger. Australian timber of ordinary proportions is not likely to fail by shearing along the fibres near the neutral axis, except from the presence of gum veins.

For ironbark timber—

$$\frac{d}{l} = \frac{1200}{18000} = \frac{1}{15} \text{ for a distributed load}^1$$

$$\text{and } \frac{1}{7 \cdot 5} \text{ for a central load.}^1$$

If the depth of the beams exceeds the proportion of the span given in the above examples, the beam may fail by longitudinal shearing along the grain; if it is less, there will be no danger of shearing, and the strength of the beam will be proportional to its modulus of rupture.

---

[1] These values of S_A and f are about correct for beams of 6 × 4 in. cross-section. For beams of 12 × 12 in. they are about 750 and 12,000 pounds per square inch respectively.

CHAPTER VI.

BRIDGES AND VIADUCTS OF SMALL SPAN IN TIMBER AND STEEL.

TIMBER PILE BRIDGES.

IN Australia, America and in other countries timber is largely used for the construction of highway and railway bridges, also for ordinary building construction. It is proposed in this chapter to give a few typical examples of the use of timber in the construction of bridges and viaducts of small span which will also serve as illustrations of the application of the principles explained in the foregoing chapters.

Figs. 155 to 159 show the method adopted by the engineers of the P.W.D., Sydney, in the design of highway bridges of Australian timber for spans of 30 to 45 ft.

The clear width of deck in each case is 15 ft., formed with planking 4 in. thick carried on four main longitudinal beams of the dimensions shown, depending on the length of the span. The superstructure, trestle piers and the abutments are all of hardwood timber. It will be observed, from inspection of Figs. 155 to 159, that the outer beams are rectangular in cross-section, and that the inner beams are tree trunks with the bark removed, merely adzed at the top or where they rest upon corbels, as shown. The piers consist each of three timber piles driven in the usual manner by a falling weight. The piles are frequently provided with shoes such as the one illustrated in Fig. 158, and they are braced together with diagonal and horizontal timber sections securely bolted as shown in Fig. 157. The beams are in every case provided with corbels at their ends over the piers, also bolts and wedges are inserted to secure the beams and corbels both horizontally and vertically. For spans of 30 ft. there is only one corbel 8 ft. long over the pier, but for spans of 45 ft. there are two corbels, one of 15 ft. and the other of 8 ft. in length. The object of the corbels is to reduce the effective span of the main beams. The strength of the 45 ft. spans will now be considered. The safe working extreme fibre dead load stress may be taken as 3000 pounds per sq. in., so that the moment of resistance of one outer beam 13 × 14 in. in cross-section is—

$$\frac{13 \times 14 \times 14 \times 3000}{6} = 1{,}274{,}000 \text{ inch-pounds,}$$

or 1274 units of 1000 inch-pounds.

The moment of resistance of one inner beam 20 in. in diameter reduced to a depth of 17 in. by cutting off a piece on the top is approximately—

$$\frac{\pi \times 20^4 \times 3}{64 \times 8} = 2937 \text{ units of 1000 inch-pounds.}$$

This is probably a little too great, but we may safely take it as 2548 units or twice as great as that of the outer beam. The load upon the inner beam is about twice as great as that upon the outer beam.

The total dead load consisting of the deck, hand rails and beams is about—

630 cub. ft. at 80 pounds = 50,400 pounds,

or, say, 50 units.

The dead load bending moment in the centre of the span is—

$$\frac{50 \times 35 \times 12}{8} = 2625 \text{ units.}$$

The effective span is assumed to be only 35 ft. in consequence of the reduction in the total span due to the two corbels.

The total load distributed over the effective span is—

$$15 \times 35 \times W = 525W$$

where W denotes the safe load per sq. ft.

The bending moment due to this load is—

$$\frac{525W \times 35 \times 12}{8} = 27{,}562W \text{ inch-pounds.}$$
$$= 27{\cdot}6W \text{ units, say.}$$

The bending moment due to the dead load and the distributed safe load on the deck is resisted by the four beams, therefore—

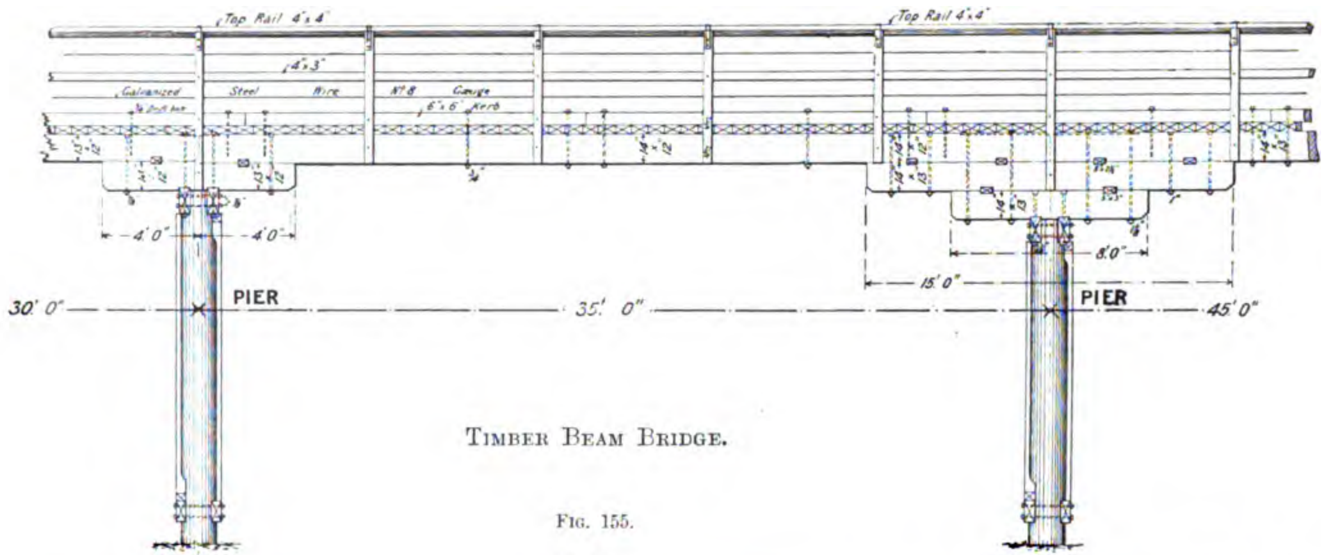
$$2625 + 27{\cdot}6W = 7644$$
$$\therefore W = 182 \text{ pounds per sq. ft. nearly.}$$

Dead Load and Traction Engine.—Consider the traction engine shown in Fig. 390 weighing 16 tons, with the most heavily loaded pair of wheels standing in the centre of the span. The maximum reaction due to 9·5 tons in the centre and 6·5 tons at 124 in. to the left of the centre is—

$$4{\cdot}75 + \frac{334}{420} \times 6{\cdot}5 = 9{\cdot}92 \text{ tons.}$$

The bending moment in the centre is—

$$9{\cdot}92 \times 210 - 6{\cdot}5 \times 124 = 1277{\cdot}2 \text{ inch-tons.}$$
$$= 2{,}860{,}928 \text{ inch-pounds.}$$
$$= 2861 \text{ units, about.}$$



TIMBER BEAM BRIDGE.

FIG. 155.

PLAN.

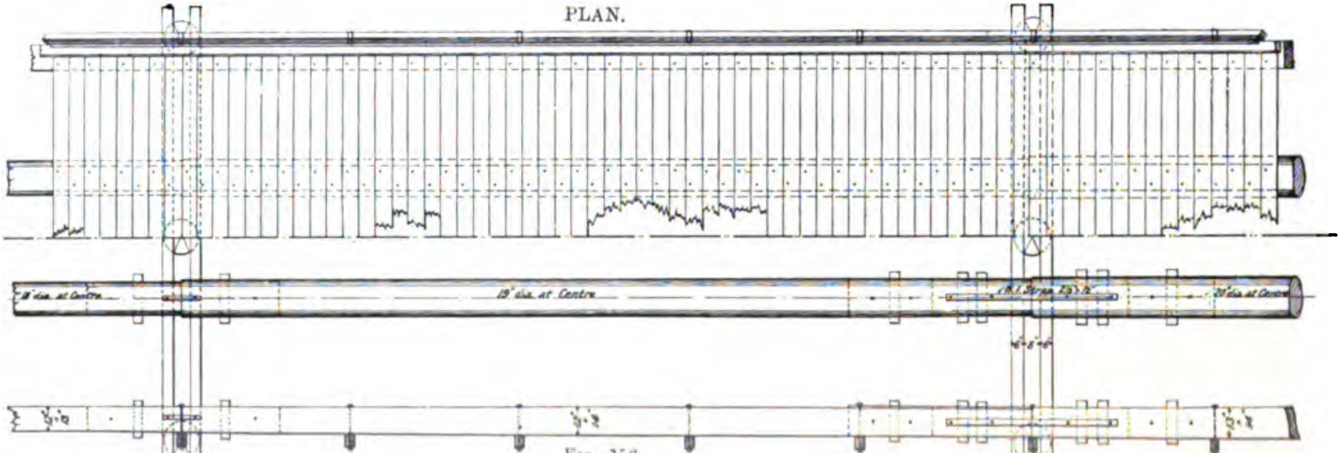


FIG. 156.

TIMBER BEAM BRIDGE.

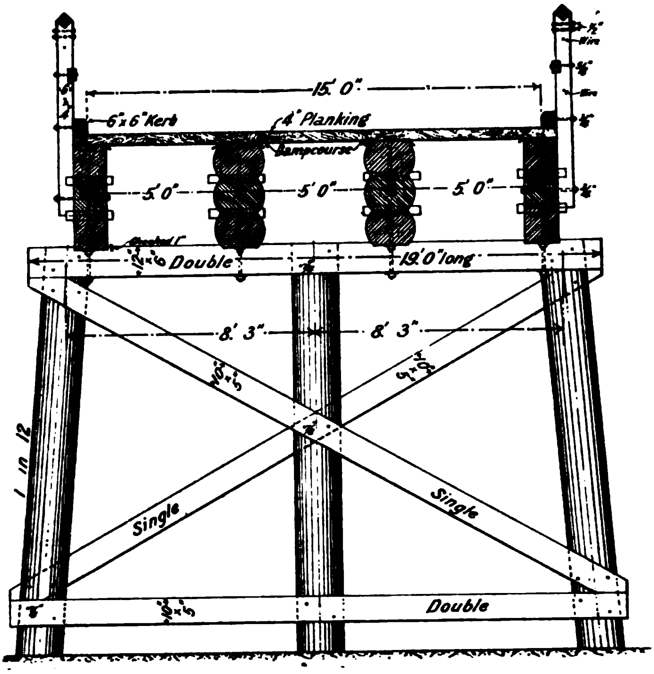
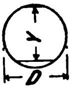


FIG. 157.

SIZES OF INNER GIRDERS

Dia. at Centre	Minimum Dia. at small end D	Minimum Circum. at small end	Dimension "Y" at small end
18"	16"	50¼	13"
19"	16"	50¼	14"
20"	16"	50¼	14"



PILE SHOE

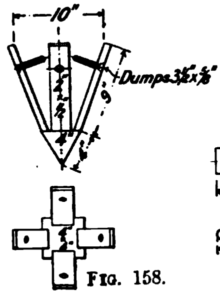


FIG. 158.

TANG BOLT
FOR ABUTMENTS

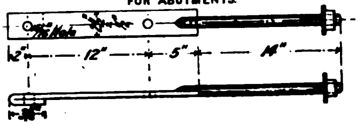


FIG. 159.

The total bending moment due to the dead load, traction engine, and impact effect is as follows:—

$$\text{Dead load} = \frac{2}{3} \times 2625 = 1750 \text{ units.}$$

$$\text{Live load} \quad . \quad . = 2861 \quad ,,$$

$$\text{Impact} = \cdot 3l\left(\frac{2861}{4611}\right) = \underline{532} \quad ,,$$

$$\text{Total equivalent dead load moment} = 5143 \quad ,,$$

The moment of resistance of the two inner beams resisting this bending moment is 5096 units, so that the bridge is barely strong enough to carry the traction engine and the dead load. These bridges, however, are not subjected to heavy traffic, and are only occasionally loaded with a traction engine.

Figs. 160 to 163 show a timber viaduct for carrying a single line of railway over low-lying land liable to floods on the New South Wales railways. The structure is built with 12 × 12 in. beams of hardwood timber on timber piles or concrete piers, spaced 6 ft. apart centre to centre. The length of the main beams is usually 24 or 32 ft., and a corbel is used over the pier at the ends of the beams. The rails are fixed directly to the tops of the main beam as shown. The effective length of any of the spans may be safely taken at 5 ft., since the corbel and the effect of continuing the beams over at least three piers reduces the effective span, as explained in Chapter XI. The maximum bending moment will occur when the most heavily loaded axle of the locomotive is on the centre of the span. The bending moment is—

$$\frac{W \times 5 \times 12}{4} = 15W \text{ inch-pounds}$$

where W denotes the effective load on the axle.

The moment of resistance of two beams each 12 × 12 in. cross-section for a safe fibre stress of 3000 pounds per sq. in. is—

$$\frac{2 \times 12 \times 12 \times 12}{6} \times 3000 = 1{,}728{,}000 \text{ inch-pounds}$$

$$\therefore W = \frac{1{,}728{,}000}{15} = 115{,}200 \text{ pounds.}$$

If we allow 100 per cent. for the impact effect of the live load, there will be more than sufficient for an axle load of 50,000 pounds and the dead load of the structure. This construction may be used with soft wood timber by modifying the sections of the beams.

6 FT. TIMBER OPENINGS.

FIG. 160.

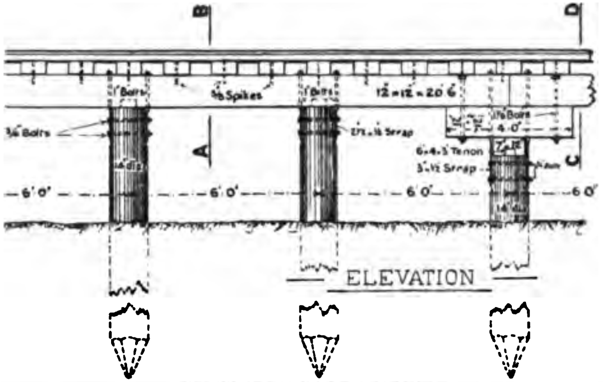


FIG. 162.

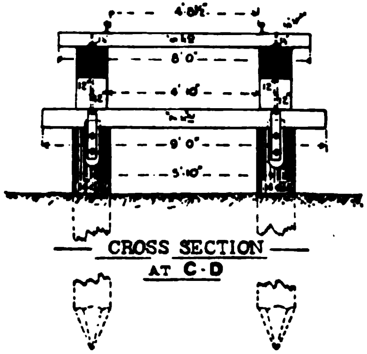


FIG. 161.

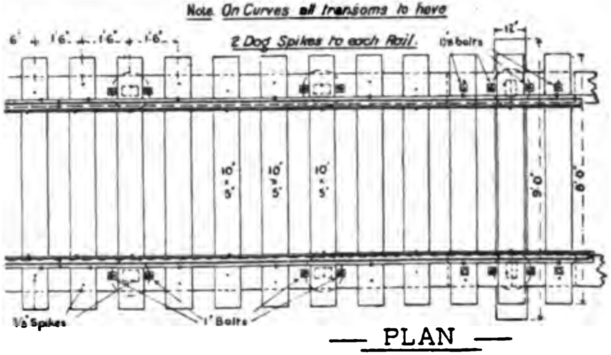
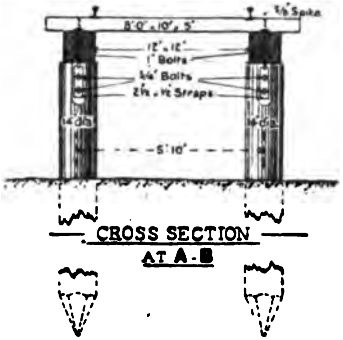


FIG. 163.



Figs. 164 and 165 illustrate the construction of a timber viaduct of hard-wood timber, as used for the heaviest traffic on the New South Wales railways. A similar construction is frequently used for highway bridges, but it is not the most economical design for pine timber, where beams of greater depth are obtainable. The design illustrated is the outcome of considerable experience in the design of viaducts of such timber as iron-bark, and numerous full-size tests have proved its efficiency. The rails are attached to cross-ties 10 × 6 in. cross-section spaced 18 in. centres, upon three compound beams arranged to equally sustain the train loads. The beams are provided with corbels at each end 9 ft. 6 in. long, carried upon timber trestle piers spaced 24 ft. centre to centre. Each compound beam is constructed with two beams of 12 × 12 in. cross-section, placed one over the other, bolted together in the manner shown, with timber keys arranged to resist the horizontal shearing stress developed in the compound beams.

The design illustrated somewhat resembles a continuous girder or cantilever type, but the compound beam is inferior to the continuous solid beam, of the same depth, both in strength and stiffness. The results of tests show that this construction is about equal to the strength of a solid beam of timber 24 in. deep, but that the deflection produced by loading is somewhat greater.

In the last example the moment of resistance of a beam of hard-wood timber was found to be for a cross-section of 12 × 12 in., 864,000 inch-pounds, so that the compound beam may be taken as equal to four times this amount for a dead load stress. The equivalent uniform live load produced by a locomotive having 35,000 pounds upon each of its four driving axles, such as Cooper's E_{35}, may be found from Table LXX., for a span of 25 feet, thus—

For maximum moments—

$$\frac{7890 \times 35}{40} = 6904 \text{ pounds per lineal ft.}$$

For maximum shear—

$$\frac{9080 \times 35}{40} = 7945 \text{ pounds per lineal ft.}$$

If we take these loads as 7 and 8 units of 1000 pounds respectively for maximum moments and shears, we have, for the bending moment—

$$M = \frac{wl^2}{8} = \frac{7 \times 24 \times 24}{8} = 504 \text{ units of 1000 foot-pounds.}$$

$$= 6048 \text{ units of 1000 inch-pounds.}$$

If we add 50 per cent.[1] for the impact effect of the live load, or 3024 units, we obtain the equivalent dead load moment, 9072 units of 1000 inch-pounds.

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[1] The increase in the deflection of these bridges produced by a locomotive at 40 miles per hour is about 30 per cent. greater than that produced by the same locomotive standing on the bridge.

—24 FT. TIMBER OPENINGS

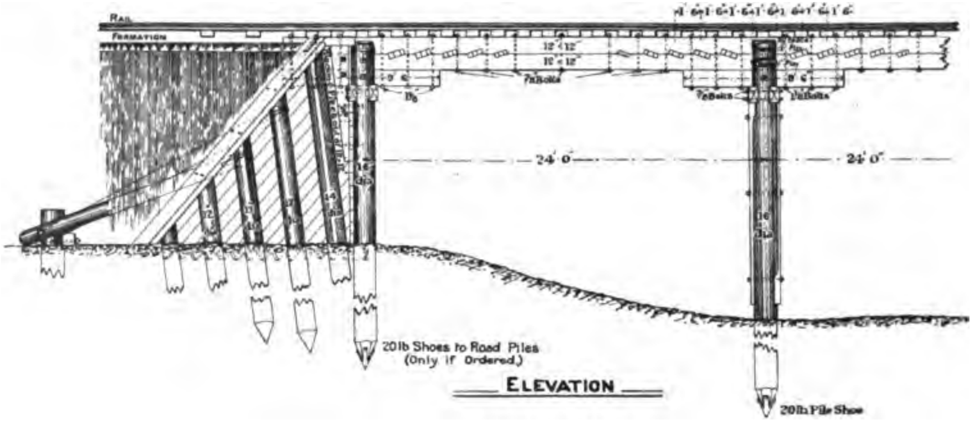


Fig. 164.

CROSS SECTIONS

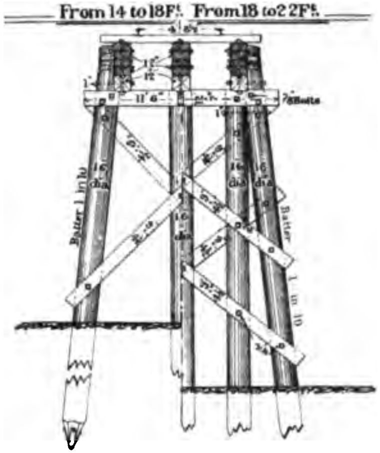


Fig. 165.

The actual dead load consisting of rails, cross-ties, beams, bolts, etc., is about 920 pounds per lineal ft., and the bending moment—

$$M = \frac{920 \times 24 \times 24 \times 12}{8} = 794{,}880 \text{ inch-pounds.}$$

$$= \text{say } 800 \text{ units of } 1000 \text{ inch-pounds.}$$

The total bending moment is—

$$9072 + 800 = 9872 \text{ units.}$$

The moment of resistance of three compound beams is—

$$864 \times 4 \times 3 = 10{,}368 \text{ units.}$$

So that they are of ample strength to resist the maximum bending moment.

The shearing stresses for the dead and live loads may be found for each compound beam, from the following equation, in units of 1000 pounds:—

$$S = 0{\cdot}307x + \frac{8}{3 \times 2 \times 24}(12 + x)^2$$

$$= 0{\cdot}307x + 0{\cdot}055(12 + x)^2$$

where x denotes the distance from the centre of the beam. The keys must be proportioned to resist the horizontal shearing stresses, and the bolts to resist the moments of these stresses tending to cause the keys to rotate and separate the two portions of the compound beam.

The following table gives the shearing stress in the keys shown in Fig. 166:—

TABLE XXXII.

x	$0{\cdot}307x$	$0{\cdot}055(12 + x)^2$	Impact effect of live load.	Total shearing stress.	Horizontal shear on keys.	Horizontal shearing-resistance provided in keys.
0	0	6·6	3·3	9·9		
3·65	1·1	13·3	6·7	21·1	37·0	43·2
5·4	1·6	16·6	8·3	26·5		
7·0	2·1	19·9	10·0	32·0	24·4	50·4
8·5	2·6	23·1	11·5	37·2	20·2	50·4
10·0	3·1	26·6	13·3	43·0	30·0	57·6
12·0	3·7	31·7	15·9	51·3	48·0	57·6

The horizontal stress upon the keys and the tensile stress upon the bolts may be considered in the following manner (Fig. 166):—

The horizontal shearing stress tends to make the two portions of the beam slide relatively to each other, and to rotate the key. This is resisted by the shearing strength of the key, and by the couple whose moment is equal to the tensile stress in the bolts multiplied by the distance between them. The forces may be assumed to act as illustrated in Fig. 166.

$$4h = 18t.$$

The force h for the second key from the pier is—

$$h = \frac{(43 + 37{\cdot}2)18}{2 \times 24} = 30 \text{ units.}$$

$$\therefore t = \frac{4 \times 30}{18} = 6{\cdot}6 \text{ units.}$$

The shearing resistance of the key, allowing a safe stress of 400 pounds per sq. in., is—

$$\frac{12 \times 12 \times 400}{1000} = 57{\cdot}6 \text{ units.}$$

The resistance of a bolt $1\frac{1}{8}$ in. diameter, allowing a safe stress of 10 units per sq. in., is—

$$10 \times 0{\cdot}69 = 6{\cdot}9 \text{ units}$$

where 0·69 is the area at the bottom of the thread in sq. in. The keys are made in the form of wedges, in three portions, and the centre wedge is driven tightly between the outside ones, in order to force them to bear hard upon the ends of the notches cut in the beams. The bolts are provided with large washers.

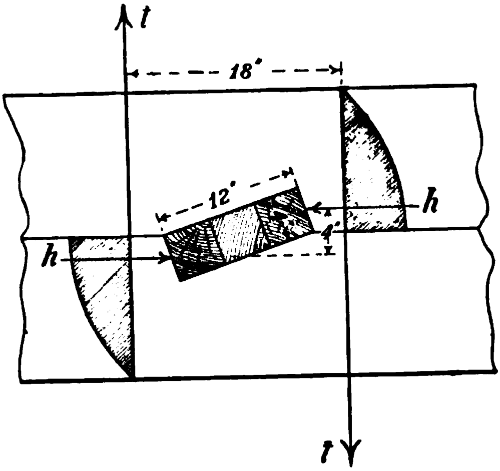


Fig. 166.

In America the open deck is used largely for railway bridges, and consists of cross-ties, or sleepers, spaced so that the distance between is never greater than 6 inches, and frequently less. If the cross-ties are 10 in. wide × 8 in. deep the distance between the centres must not exceed 16 in., they are notched down $\frac{1}{2}$ in. and generally every fourth tie is bolted to the supporting girders with $\frac{3}{4}$ in. bolts. In deck bridges the cross-ties extend the full width between the outside edges of the main girders, and in through bridges every other tie extends the full width for a footpath. On each side of the track guard timbers are provided notched from $\frac{3}{4}$ to 1 in. over every cross-tie, bolted to every third tie with $\frac{3}{4}$ in. bolts, and also through the lap splices where they join. It may be safely assumed that the live load on one cross-tie is equal to one-third of the load upon a driving axle. If the centres of the supporting girders are spaced 7 ft. apart, the maximum bending moment in the tie will be for an axle load of 50,000 pounds—

$$\frac{50{,}000 \times 12}{6} = 100{,}000 \text{ inch-pounds.}$$

The moment of resistance of the ties 10 × 8 allowing 1000 pounds per sq. in. for the extreme fibre stress, as recommended by Mr. Cooper for yellow pine and white oak, is—

$$\frac{10 \times 64 \times 1000}{6} = 106{,}000 \text{ inch-pounds.}$$

For highway bridges a fibre stress of 12,000 pounds per sq. in. is permitted.

Figs. 167, 168 and 169 illustrate the standard pile-trestle on the Toledo, St. Louis and Kansas City Railroad.[1] The sleepers are 9 ft. long × 8 in. wide × 6 in. deep, and the guard-rails 6 in. × 6 in., notched 1 in. over sleepers. The main beams or stringers are arranged in groups of three under each of the rails, and are 7 in. wide × 16 in. deep on a span of 15 feet.

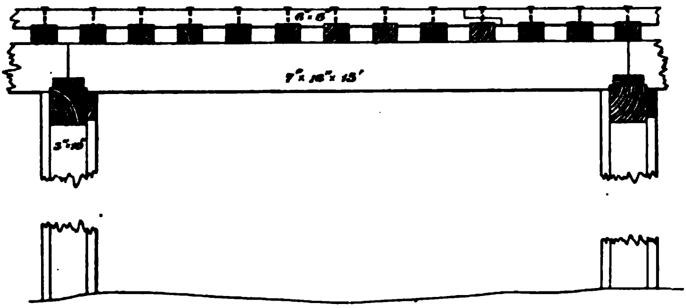


FIG. 167.

Cast-iron separators 4 in. thick are arranged between the stringers over the caps, and the stringers are protected with sheet-iron 30 in. wide, as shown in Fig. 168. The caps are 14 ft. long × 12 in. × 12 in., notched 1 in. over the four piles, and the sway braces are 10 in. × 3 in. On the Pennsylvania Railroad the sizes of the stringers used are given in the following table :—

---

[1] *Treatise on Wooden Trestle Bridges*, by Wolcott C. Foster. Published by J. Wiley & Sons, New York.

TABLE XXXIIA.

Clear span in feet.	Number of stringers under each rail.	Width of each stringer in inches.	Depth of stringer in inches.
10	2	8	15
12	2	8	16
14	2	10	17
16	3	8	17

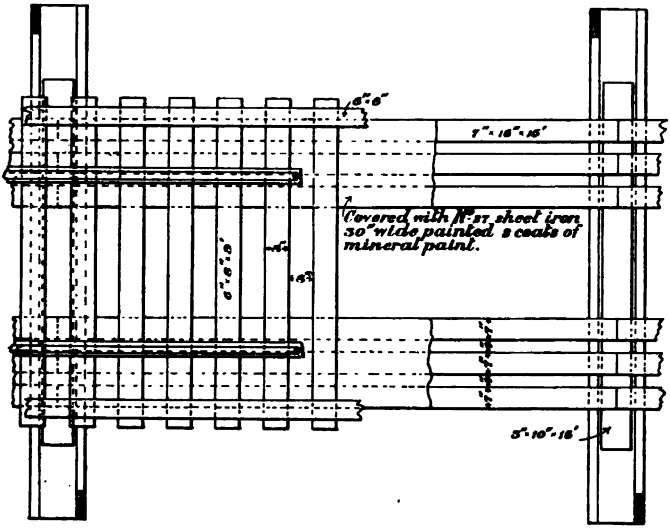

6"×6"
7"×16"×16'
Covered with No. 27 sheet iron 30" wide painted 2 coats of mineral paint.
6"×8"×9'
5"×10"×16'

FIG. 168.

The stringers are generally long enough to extend over two trestles, and should break joint, and should be securely fastened to the caps by means of drift-bolts.

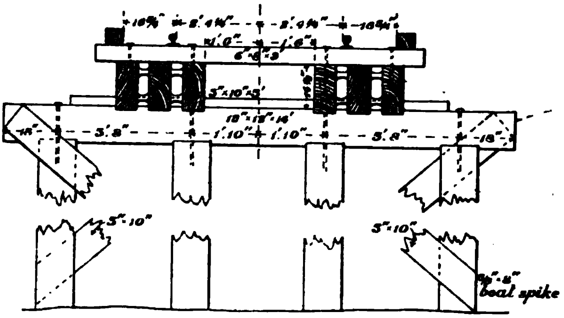


FIG. 169.

Fig. 170 shows the method of connecting the stringers over the caps adopted in the Pennsylvania Railroad, in which the tendency of the joint, when it settles under a weight through the support becoming weakened, is to close at the top and to open at the bottom. This arrangement provides most material where there is the greatest liability to split. The packing strips are notched over the caps as shown. On the Chicago, Milwaukee and St. Paul Railroad the standard pile-trestle in 1890 had six stringers in two groups of three, each 16 in. deep × 8 in. wide, on a span of 15 ft. 9 in., and two outside stringers each 16 in. deep × 6 in. wide. The dimensions of the stringers will depend upon the character of the timber available, and the train loads.

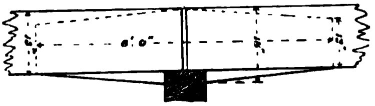


Fig. 170.

With an equivalent uniform load of 3·5 tons per foot-run allowing for the dynamic effect of the live load, and stringers 16 in. × 8 in. on a span of 15 ft., we have—

$$\text{Bending moment} = \frac{3{\cdot}5 \times 15 \times 15}{8} = 98{\cdot}43 \text{ foot-tons,}$$

$$= 1181 \text{ inch-tons.}$$

The moment of resistance of the six beams is—

$$\text{Moment of resistance} = \frac{6 \times 8 \times 16 \times 16}{6} f = 2048 f.$$

For white pine or spruce, Prof. Lanza recommends $f = 3000$, and the experiments of Mr. Onward Bates on white pine stringers[1] show that this value is about correct; hence—

$$\text{Moment of resistance} = 2048 \times 3000 = 6{,}144{,}000 \text{ inch-pounds,}$$

$$= 2743 \text{ inch-tons.}$$

Therefore the factor of safety would be—

$$\frac{2743}{1181} = 2{\cdot}3.$$

For yellow pine $f = 5000$, and the factor of safety is 3 83. For Oregon fir $f = 6500$, and the factor of safety is 5·4. The latter value of the factor of safety may be considered sufficient, but it is clear that American engineers do not use large factors of safety in their timber structures.

There are two distinct classes of timber trestle piers—(*a*) framed trestles; (*b*) pile trestles.

Framed trestles consist of square timber framed together; they are usually built upon a foundation of some kind, which may be a pile foundation, masonry, concrete, grillage, crib, solid or loose rock.

---

[1] *Trans. Amer. Soc. C. E.*, November, 1890.

Framed trestles are provided with a sill which rests upon the foundations, and, in order that decay may be delayed as long as possible, the sill should be above the ground, and not covered with earth.

Framed trestles are used for almost any height, but they are exclusively used for lofty timber viaducts.

Pile trestles consist of two or more piles surmounted by a cap. When the height is less than 10 feet sway bracing is not used; above this height sway braces are used, bolted to the cap and to the piles at their intersection. Pile trestles rarely exceed 30 ft. in height; from 10 to 20 ft. the outside piles are driven with a batter of from 1 to 3 in. per ft.

The timber used for piles should be straight and sound, free from wind-shakes, wanes, large loose black or decayed knots, cracks, wormholes, and all descriptions of decay, and should be stripped of bark; they must be cut off square at the butt, and be properly sharpened.

Piles of Australian hardwood, which is liable to decay at the heart, should be free from large pipes.

The heads and feet of piles should be protected before driving, if they are liable to be injured by the hammer or ram. A wrought-iron ring is generally used at the head, and some form of iron shoe at the foot according to the nature of the material into which the pile is driven. Timber piles for viaducts are usually driven by a drop-hammer or ram; other methods will therefore not be considered. An upright frame is used with a pulley at the top, over which a rope passes which supports the hammer or ram. The frame consists of two uprights called leaders, from 10 to 60 ft. long, placed about 2 ft. apart, which guide the falling ram. The weight of the ram varies from 500 to 7000 lbs., but is usually about one ton, and it is provided with grooves or guides to fit the guides. The rope is usually wound up and the weight raised by a steam engine, but horses may be used hitched directly to the end of the rope, or men may be employed working a windlass or pulling directly at the rope. There are two methods of detaching the weight, *i.e.*, letting the ram fall (*a*) by means of a nipper; (*b*) by means of a friction clutch.

A pile is considered to be sufficiently driven if the penetration from the last five blows with a ram weighing 2000 lbs. falling 25 ft. does not exceed 5 in. Heavier rams with smaller falls give better results, such as at Brooklyn, where a ram weighing 6720 lbs. was used falling 3 ft.

The bearing power of piles may be considered in the following manner: When a pile rests upon a hard stratum it should be treated as a long column, and its bearing capacity may be inferred by means of the tables, pp. 60, 61 and 64, 65, Chapter II., or it may be calculated by means of formulæ such as those given in Chapter XII. The safe bearing pressure on a pile driven into a yielding stratum may be found in the following manner:—

(*a*) By considering the relationship between the supporting power and the length and size of the pile, the weight of the ram, height of fall, and the distance the pile was moved by the last blow. The pile is assumed to be driven under ordinary conditions; the head, if broomed or battered unreasonably, should be sawn off before striking the test blow; and it is also assumed that the pile has been sinking with a fair degree of regularity under the last few blows, and that the apparent uniformity of set is not deceptive.

(*b*) By applying a load or direct pressure to each of a number of piles, observing the amount each will support, and expressing the result in terms of the depth driven, size of pile, and kind of soil. The former method may be expressed in a rational formula, the latter in an empirical formula.

The former method has been used more or less in the formulæ of Rankine, Weisbach, Sanders, Prof. Baker, Trautwine, Crowell, Wellington and others.

The energy accumulated in the ram when it strikes the pile head is Wh, where W = weight of ram in pounds, and h = height of fall in feet. This energy is expended in compressing the ram and the head of the pile, in moving the ram as a whole against the resistance of the soil, in overcoming the inertia of the pile and soil. Only Wellington's formula will be here given, as it is a reliable formula less complex than the others, and gives quite as good results.

Let L denote the bearing pressure; s, the penetration under the last blow in inches; then—

$$L = \frac{12Wh}{s+1}.$$

Wellington recommends a factor of safety of 6, so that the safe working pressure, L′, is—

$$L' = \frac{2Wh}{s+1}.$$

Bolts, Washers, Spikes, etc.—The bolts used in holding stringer pieces together, fastening on the sway-braces, guard-rails, etc., are usually made ¾ in. in diameter. In compound timber beams the bolts may be from 1 in. to 2 in. in diameter, being proportioned to resist the shearing stresses in the manner explained. Cast- or wrought-iron washers should be placed under the heads and nuts of all the bolts in a timber structure. Sometimes the washers under the nut are provided with slots, which enable the nut to be locked by driving in a nail close to the nut after it has been screwed down tightly.

Cast-iron washers for bolts ¾ in. to 1¼ in. in diameter vary from 3 in. to 4½ in. in diameter by ¾ in. thick. Drift-bolts are largely used in timber structures, such as in fastening caps to posts; they are usually made of iron, ¾ in. square or ¾ in. round. They should always be long enough to penetrate the last timber which it is desired to hold to a depth sufficient to give a good firm hold. Drift-bolts have square or taper heads, and are pointed at the ends; they are really long spikes.

Experiments made by the United States Government show that smooth rods have a greater holding power than ragged ones; that the resistance ten months after being driven is 10 per cent. greater than the resistance immediately after driving; that round drift-bolts are 25 per cent. more efficient per pound of metal than square ones. The holding power of a round bolt 1 in. in diameter in a ¾-in. hole, with white pine and hemlock timber, is about 10,000 pounds per lineal ft. immediately after driving. With Norway pine the resistance is about 9000 pounds per lineal ft.

The resistance of screw-bolts was found to be about 50 per cent. greater than that of plain round rods. All bolt-holes in pine timbers should be bored ⅛ in. smaller than the bolt. The holding power of drift-bolts or dog-spikes in Australian hard-wood timbers is much greater than in pine timbers; but the actual resistance developed has not been determined, as far as the author is aware.[1]

Painting Timber.—Whenever two surfaces of timber touch, they should be painted with white lead or tar; the bolts should also be coated with white lead and linseed oil, hot or cold tar. Mr. Wolcott C. Foster recommends[2] the following specification for creosoted trestles, piles, and other timber:—

"All piles used in creosoted trestles must have the bark peeled off, and be pointed before treatment. None of the sap wood must be hewn from the piles. No notching or cutting of the piles will be allowed after treatment, except the sawing off of the head of the pile to the proper level for the reception of the cap, and the levelling of such part of the head as may project from under the cap.

"The heads of all creosoted piles, after the necessary cutting and trimming has been done to receive the cap, must be saturated with hot creosote oil, and then covered with hot asphaltum before putting the caps in place.

"Timber in creosoted trestles must be cut and framed to the proper dimensions before treatment. No cutting or trimming of any kind will be allowed after treatment, except boring of the necessary bolt-holes. Hot creosote oil must be poured into the bolt-holes before the insertion of the bolts, in such a manner that the entire surface of the holes shall receive a coating of creosote oil. All creosoted timber and piles shall be prepared in accordance with the following process:—

---

[1] This matter is now being investigated in the Author's laboratory.

[2] *A Treatise on Wooden Trestle Bridges.*

"The timber and piles, after having been cut and trimmed to the proper length, size, and shape, shall be submitted to a contact steaming inside the injection-cylinders which shall last from two to three hours, according to the size of the timbers; then to a heat not to exceed 230° Fahr. in a vacuum of 24 in. of mercury, for a period long enough to thoroughly dry the wood. The creosote oil, heated to a temperature of about 175°, shall then be let in the injection-cylinder, and forced into the wood under a pressure of 150 pounds per sq. in., until not less than 15 pounds of oil to the cubic ft. has been absorbed.

"The oil must contain at least 10 per cent. of carbolic and cresylic acids, and have at least 12 per cent. of naphthalin."

Small Flood Opening.—Figs. 171 to 173 illustrate a small flood opening on a single line of railway, which is constructed with four

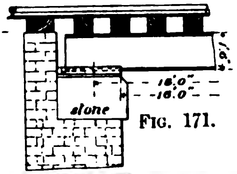


FIG. 171.

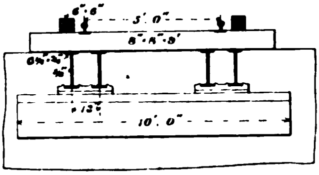


FIG. 173.

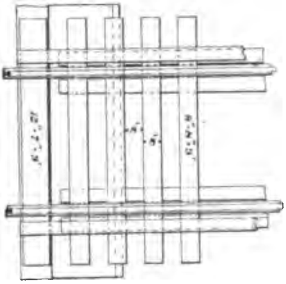


FIG. 172.

rolled steel girders resting on cast-iron bed-plates attached to the coping stone of the brick supports. The span is 16 ft. in the clear; and 18 ft. effective between centres of bearings. The rolled girders are 18 in. deep and weigh 78 pounds per foot run (see Table XXXIA). If we allow 17,000 pounds per sq. in. for a dead load stress, or 17 units of 1000 pounds each, the moment of resistance of the four girders is—

$$\frac{fI}{y} = 4 \times 128{\cdot}6 \times 17 = 8744{\cdot}8, \text{ say } 8745 \text{ units of } 1000 \text{ inch-pounds.}$$

The bending moment produced by a standard engine load known as Cooper's E_{40} (Fig. 378), may easily be shown to be 3840 units of

1000 inch-pounds. The dead load consists of the deck and four rolled girders, which may be taken at—

$$400 + 4 \times 78 = 712 \text{ pounds per ft. run.}$$

The total dead load is therefore 12,816 pounds, or say 13 units, and the central bending moment is 351 units of 1000 inch-pounds. The equivalent dead load bending moment, allowing 100 per cent. for the impact effect of the engine, is—

$$3840 + 3840 + 351 = 8031 \text{ units.}$$

The four girders selected are therefore of ample strength for the purpose, and the design illustrated in Figs. 171 to 173 represents a simple and economical bridge for a small span.

CHAPTER VII.

EXAMPLES OF GRAPHICAL STATICS APPLIED TO THE DETERMINATION OF THE STRESSES IN TRUSSES.

THE graphical method of determining the stresses in structures described in Chapter III. may be applied to any form of truss, and also to the solution of a great variety of problems in connection with structures and machinery. The method will be used in this chapter in connection with a few ordinary trusses such as are used for roofs; but it will generally be advisable in such cases to apply also the method of moments in order to check the work, measuring the lever arms of the forces from the same scale drawing of the truss which is used in connection with the reciprocal diagrams of stresses.

It is a good plan to tabulate the stresses obtained by measuring the lines in the reciprocal diagrams side by side with those obtained by the method of moments, and to use the mean of the two stresses so obtained in determining the total maxima stresses from which the structure is designed.

The amount and distribution of the load upon roofs will be first considered.

Dead Load upon Roofs.—The dead load upon roofs consists of the weight of the roof principals or trusses; the purlins, which are the longitudinal beams carrying the load between the trusses and discharging it on the principals; also the rafters and the roof-covering.

The weight of the roof principal or truss may be estimated from the known weight of similar roofs for the purposes of calculation, and the sectional areas thus determined; afterwards the correct weight may be ascertained, and the truss recalculated with the corrected weight.

The weights of the purlins may be ascertained without much trouble, as they consist of timber beams, angles, T's, or rolled girders. In large roofs the purlins may be lattice girders or truss-beams.

The weight of the roof-covering may be estimated by making use of the following table, which is taken from *Instruction in Construction*, by Colonel Wray, R.E. :—

TABLE XXXIII.

WEIGHT OF ROOF-COVERING.

Description of material.	Weight in pounds per square foot.
Lead covering, including laps, but not boarding or rolls	5½ to 8½
Zinc covering, including laps 14 to 16 zinc gauge	1½ to 1¾
Corrugated iron, 16 W.G.	3½
" " 18 "	2½
" " 20 "	2
Sheet iron, 16 W.G.	2½
" " 20 "	1½
Slating laid with a 3-inch lap, including nails, but not battens or iron laths:—	
Slates, doubles, 13 in. × 9 in., at 18 cwt. per 1200	8¼
Ladies, 16 in. × 8 in., at 31·5 cwt. per 1200	8¼
Countesses, 20 in. × 12 in., at 50 cwt. per 1200	8
Duchesses, 24 in. × 12 in., at 77 cwt. per 1200	8½
Slate battens, 3½ in. × 1 in.:—	
For doubles	2
For countesses	1¼
Boarding, ¾ in. thick	2½
" 1 " "	3½
" 1¼ " "	4¼
Wrought-iron laths, angle-irons:—	
For duchess slates	2
For countess slates	1¾

The weight of snow to be allowed for depends entirely upon the locality. In England it is sometimes assumed that a roof of flat pitch may have 6 in. of snow upon it, the weight of which may be 5 lbs. per sq. ft.

Trautwine, in his handbook, recommends an allowance of 12 lbs. per sq. ft. of area covered; and Stoney, 20 lbs. per sq. ft. The depth of snow will diminish as the pitch of the roof increases.

The wind pressure is estimated in the manner explained in Chapter XIV.

The following table gives the weights of some well-known roofs:—[1]

---

[1] *Notes on Building Construction*, by Rivington, Part IV.

TABLE XXXIV.

Description of roof.	Clear span in feet.	Distance apart of principals in feet.	Weight in pounds per square foot of area covered.			
			Of purlins.	Of principal.	Total ironwork.	Total with covering.
Common truss	37	5	1·1	3·5	4·6	6·9
	40	12	2·0	3·5	5·5	—
	54	14	6·5	3·0	9·5	—
	55	6½	4·6	7·0	11·6	—
	72	20	4·2	2·8	7·0	—
	84	9	2·6	5·9	8·5	—
	50	10	—	—	3·0	5·2
	100	14	—	—	7·0	9·0
	130	26	0·8	5·6	6·4	8·0
	140	12	—	4·5	—	—
Bowstring roof, Manchester, London Road Station	50	11	—	—	9·6	—
Bowstring roof, Lime Street Station, Liverpool	154	26	—	4·9	—	—
Bowstring roof, Birmingham	211	24	—	7·3	11·0	—
Arched roofs, corrugated iron	40	—	—	—	—	2·5
	60	—	—	—	—	3·5
Strasburg Railway	97	13	—	—	12·0	—
Paris Exhibition	153	26	9·5	5·5	15·0	—
Dublin	41	16	3·4	7·3	10·7	—
Derby	81½	24	10·8	6·0	16·8	—
Sydenham	120	—	7·9	3·9	11·8	—
St. Pancras	240	29½	7·4	17·1	24·5	—

The following table, taken from Trautwine's handbook, may be used in estimating approximately the total loads on roof-trusses up to 75 ft. span; the principals or trusses are supposed to be spaced 7 ft. apart, centre to centre.

TABLE XXXV.

TABLE OF APPROXIMATE LOADS ON ROOFS IN POUNDS.

Description of roof.	Weight of roof, or dead load.	Wind and snow.	Total loads.
Roof covered with corrugated iron weighing from 1½ to 2 lbs. per square foot, unboarded	8	20	28
Ditto, if plastered below rafters	18	20	38
Ditto, corrugated iron on boards	11	20	31
Ditto, if plastered below rafters	21	20	41
Ditto, slate unboarded or on laths	13	20	33
Ditto, „ on boards 1¼ in. thick	16	20	36
Ditto, „ if plastered below rafters	26	20	46
Ditto, „ shingles on lath	10	20	30
Ditto, „ if plastered below rafters or tie-beam	20	20	40

For spans from 75 to 150 ft., it will suffice to add 4 lbs. to each of the above total loads.

The above total loads may be considered to act vertically, and a reciprocal diagram similar to those obtained for the dead load may be drawn to determine the stresses and the dimensions of the various members, from which the weight of the truss or principal may be calculated. The weight thus obtained may be used in estimating the dead loads for the complete calculations.

In Chapter III. two methods were considered for determining the stresses in braced structures, which were illustrated with reference to the common roof-truss shown in Fig. 174. The stresses in this roof will now be more fully considered, and, in the first place, the results obtained in Chapter III. may be tabulated thus:—

TABLE XXXVI.

DEAD LOAD STRESSES.

Bar.	Method of moments.	Scale measurement of reciprocal diagram.	Mean of the two methods.
	lbs.	lbs.	lbs.
AB	- 1,504	- 1,596	- 1,550
BC	+ 6,246	+ 6,240	+ 6,243
CD	+ 6,246	+ 6,240	+ 6,243
DE	- 1,504	- 1,596	- 1,550
AH	- 11,938	- 11,944	- 11,941
BI	- 11,328	- 11,296	- 11,312
JD	- 11,328	- 11,296	- 11,312
EK	+ 11,938	+ 11,944	+ 11,941
AZ	+ 11,274	+ 11,216	+ 11,247
CZ	+ 5,270	+ 5,268	+ 5,629
EZ	+ 11,274	+ 11,216	+ 11,245

With regard to the stresses due to wind:—

If a horizontal force of wind of 50 lbs. per sq. ft. act upon one side of the roof, the normal pressure due to this force may be obtained from Table LXVI., p. 316.

The angle of the roof is 22½°, so that the normal pressure is 25 lbs. per sq. ft.

Since the roof principals are 12 ft. apart centre to centre, and the length of the rafters is 21·6 ft., the total wind force on one side acting normally to the roof is—

$$12 \times 21{\cdot}6 \times 25 = 6480 \text{ lbs.}$$

If the roof principal is bolted down at each shoe, so that the ends are fixed, the stresses may be obtained by measuring the reciprocal diagram, Fig. 176. The results are given in the following table, compared with those obtained by the method of moments, thus:—

Table XXXVII.

Stresses due to Wind, both ends fixed.

Bar.	Wind on left.			Wind on right.		
	Method of moments.	Scale measurement of reciprocal diagram.	Mean of the two methods.	Method of moments.	Scale measurement of reciprocal diagram.	Mean of the two methods.
	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.
AB	- 3,240	- 3,240	- 3,240	0	0	0
BC	+ 10,377	+ 10,296	+ 10,336	+ 2,090	+ 2,280	+ 2,185
CD	+ 2,090	+ 2,280	+ 2,185	+ 10,377	+ 10,296	+ 10,336
DE	0	0	0	- 3,240	- 3,240	- 3,240
AH	- 14,574	- 14,472	- 14,523	- 7,860	- 7,974	- 7,917
BI	- 14,574	- 14,472	- 14,523	- 7,860	- 7,974	- 7,917
JD	- 7,860	- 7,974	- 7,917	- 14,574	- 14,472	- 14,523
EK	- 7,860	- 7,974	- 7,917	- 14,574	- 14,472	- 14,523
AZ	+ 14,864	+ 14,784	+ 14,824	+ 6,668	+ 6,764	+ 6,718
CZ	+ 4,775	+ 4,764	+ 4,770	+ 4,775	+ 4,764	+ 4,770
EZ	+ 6,668	+ 6,764	+ 6,718	+ 14,864	+ 14,784	+ 14,824

We see that if the wind is on the left, the bar DE is unstressed; if the wind is on the right, AB is unstressed.

The direction of the reactions R_1 and R_2 is normal to that of the principal rafter, and the values of these reactions may be determined by means of the polar and funicular polygons shown in dotted lines, Figs. 174 and 176. The point Z is determined by drawing the line OZ parallel to the closing line in the funicular polygon.

If one end of the principal is fixed, and the other end is free to move on expansion rollers, the principal will tend to expand or contract when the temperature changes, motion taking place as soon as the forces due to changes of temperature are sufficient to overcome the frictional resistance of the expansion rollers. When the frictional resistances just balance the forces due to temperature, the whole of the horizontal component of the wind pressure must be resisted by the fixed end, since the roller end can only supply a vertical reaction.

The vertical components of the normal reactions at the fixed and free ends remain the same as in the case last considered. Two cases require to be considered, namely, the wind on the left and the wind on the right (Figs. 177 and 178); wind on the left with the left end free to move (Fig. 177).

The diagram obtained for this case differs slightly from Fig. 176, as the horizontal component of the reactions at both ends must be

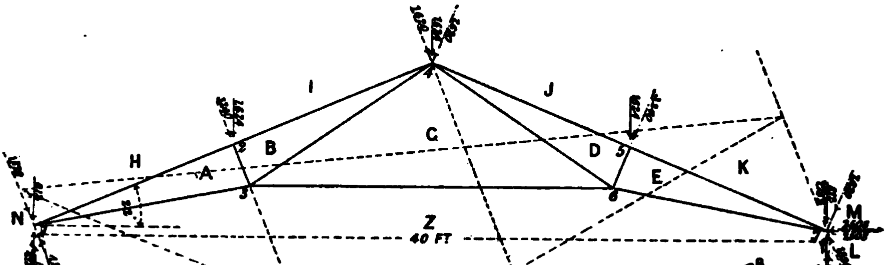


FIG. 174.

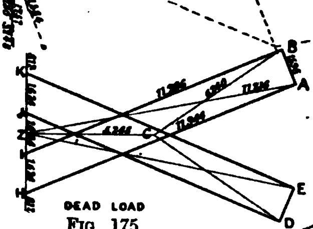


FIG. 175.

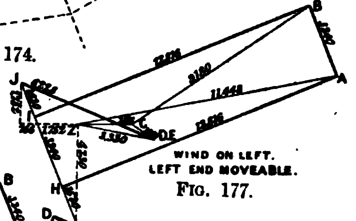


FIG. 177.

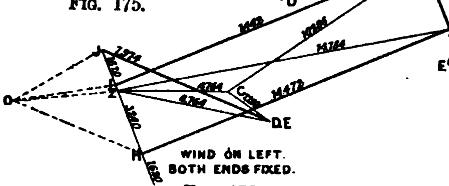


FIG. 176.

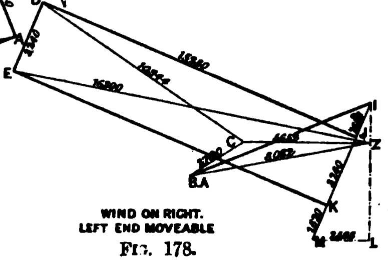


FIG. 178.

entirely supplied by the fixed end. The stresses are given in the following table:—

TABLE XXXVIII.

STRESSES DUE TO WIND ON LEFT, LEFT END MOVABLE.

Bar.	Method of moments.	Scale measurement of reciprocal diagram, Fig. 177.	Mean of the two methods.
	lbs.	lbs.	lbs.
AB	− 3,240	− 3,240	− 3,240
BC	+ 9,158	+ 9,180	+ 9,169
CD	+ 947	+ 1,098	+ 1,022
DE	0	0	0
AH	− 12,964	− 12,816	− 12,890
BI	− 12,964	− 12,816	− 12,890
JD	− 6,137	− 6,228	− 6,182
EK	− 6,137	− 6,228	− 6,182
AZ	+ 11,627	+ 11,448	+ 11,537
CZ	+ 2,478	+ 2,376	+ 2,424
EZ	+ 3,171	+ 3,330	+ 3,250

Wind on the right, with the left end movable. The stresses due to this case may be tabulated in a similar manner to the foregoing, thus:—

TABLE XXXIX.

STRESSES DUE TO WIND ON RIGHT, LEFT END MOVABLE.

Bar.	Method of moments.	Scale measurement of reciprocal diagram, Fig. 178.	Mean of the two methods.
	lbs.	lbs.	lbs.
AB	0	0	0
BC	+ 2,692	+ 2,700	+ 2,696
CD	+ 10,917	+ 10,944	+ 10,930
DE	− 3,240	− 3,240	− 3,240
AH	− 8,523	− 8,572	− 8,547
BI	− 8,523	− 8,572	− 8,547
JD	− 15,360	− 15,280	− 15,320
EK	− 15,360	− 15,280	− 15,320
AZ	+ 8,051	+ 8,082	+ 8,066
CZ	+ 5,646	+ 5,652	+ 5,649
EZ	+ 16,506	+ 16,300	+ 16,403

The details of the calculations by the method of moments is left for the student to verify.

The maximum stresses on any member of the truss due to wind may be obtained from an inspection of the foregoing tables. The resultant stress is found by combining the maximum wind stress with the dead load stress, thus:—

TABLE XL.

TABLE OF MAXIMUM STRESSES DUE TO DEAD LOAD AND WIND PRESSURE.

Bar.	Dead load stresses in pounds.	Wind stresses in pounds.	Resultant stress due to dead load and wind in pounds.
AB	− 1,550	− 3,240	− 4,790
BC	+ 6,243	+ 10,336	+ 16,579
CD	+ 6,243	+ 10,930	+ 17,173
DE	− 1,550	− 3,240	− 4,790
AH	− 11,941	− 14,523	− 26,464
BI	− 11,312	− 14,523	− 25,835
JD	− 11,312	− 15,320	− 26,630
EK	− 11,941	− 15,320	− 27,261
AZ	+ 11,245	+ 11,537	+ 22,784
CZ	+ 5,260	+ 5,649	+ 10,918
EZ	+ 11,245	+ 16,403	+ 27,648

The resultant stresses should be used in designing the truss.

The method of obtaining the stresses in a framed structure, such as a crane, is illustrated in Figs. 179 and 180.

The force polygon in this case consists simply of the line YZ, representing the load W suspended from the end of the cantilever.

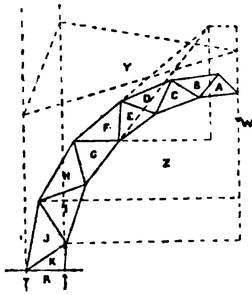


Fig. 179.

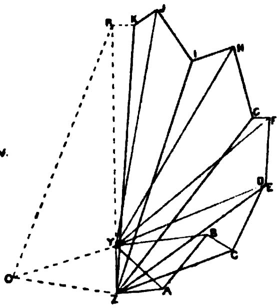


Fig. 180.

The magnitude of the reactions RY and RZ, acting as shown in Fig. 179, are found by means of the polar and funicular polygons shown in dotted lines, the point Y being determined by drawing OY parallel to the closing line as before. The magnitude and sense of the stresses in the various members of the cantilever may be determined by measuring the lines of the reciprocal diagram, Fig. 180, to the proper scale, or by applying the method of moments.

Fig. 181 represents a diagram of a roof-truss, for which the reciprocal diagrams have been drawn for the dead load and wind pressure shown in Figs. 182 to 185.

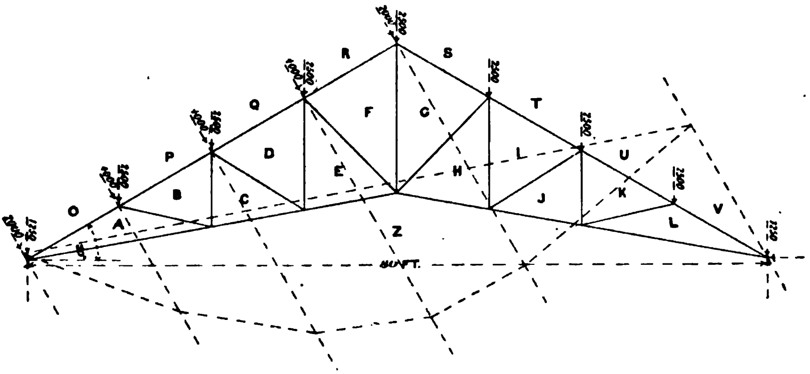


Fig. 181.

The following data have been employed in drawing these diagrams:—

Span of roof-truss	80·00 ft.
Distance between principals	10·00 „
Length of principal rafters	46·20 „
Distance between joints measured along principal rafters	11·55 „
Inclination of principal rafters to the horizon	30 deg.
Rise of tie-rod in the centre	7 ft.
Dead load at each of the seven intermediate joints	2500 lbs.
Horizontal wind pressure per sq. ft.	50 „
Normal „ „ „	33 „
Total normal pressure on one side	16,000 „
Normal pressure at the three intermediate joints on one side	4000 „
Normal pressure at the apex and point of support	2000 „

Fig. 182 represents the stresses due to the dead load; Fig. 183, the stresses due to the wind acting on the left when both ends are fixed; Fig. 184 represents the stresses due to the wind acting on the left when the left end is free to move; and Fig. 185 represents the stresses with the wind on the right when the left end is free to move.

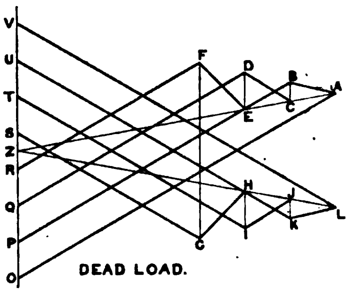


FIG. 182.

The stresses may be tabulated in a similar manner to that illustrated in connection with the roof-truss shown in Fig. 174.

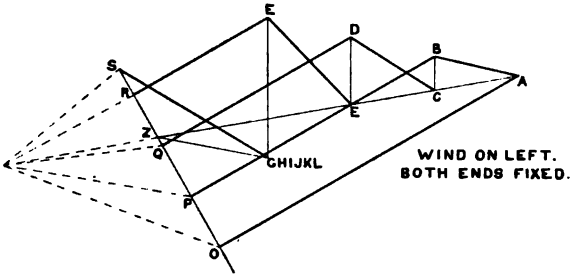


FIG. 183.

Fig. 186 represents another example of a roof-truss, for which reciprocal diagrams have been drawn from the following data:—

Span of truss	60 ft.
Distance between trusses	7·5 „
Length of principal rafters	32·5 „
Distance between the joints measured along the principal rafters	8·1 „
Inclination of the principal rafter to the horizon	22½ deg.
Rise of the main tie-rod in the centre	2 ft.
Dead load at each of the seven intermediate upper joints	1053 lbs.
Dead load at supports	526·5 „

Horizontal wind pressure per sq. ft.	**50**	**lbs.**
Normal „ „ „	**33**	**„**
Total normal pressure on one side	**8100**	**„**
Normal pressure at the three intermediate joints on one side	**2025**	**„**
Normal pressure at apex and support	**1012·5**	**„**

Fig. 187 represents the stresses due to the dead load; Fig. 188, the stresses due to the wind acting on the left when both ends are fixed; Fig. 189, the stresses due to wind acting on the left when the left end is free to move; and Fig. 190, the stresses with the wind acting on the right when the left end is free to move.

The stresses in this and the foregoing example may be tabulated in the manner fully explained in the first example, Figs. 174 to 178.

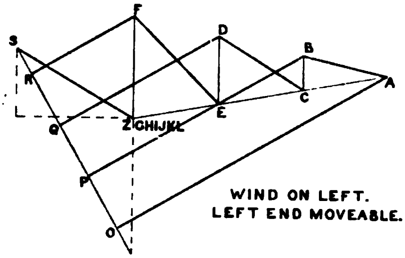


Fig. 184.

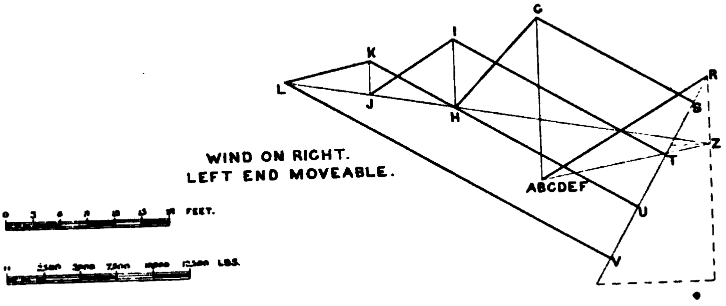


Fig. 185.

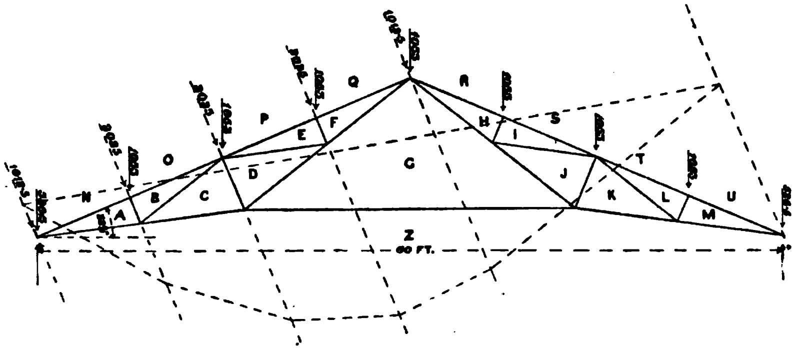
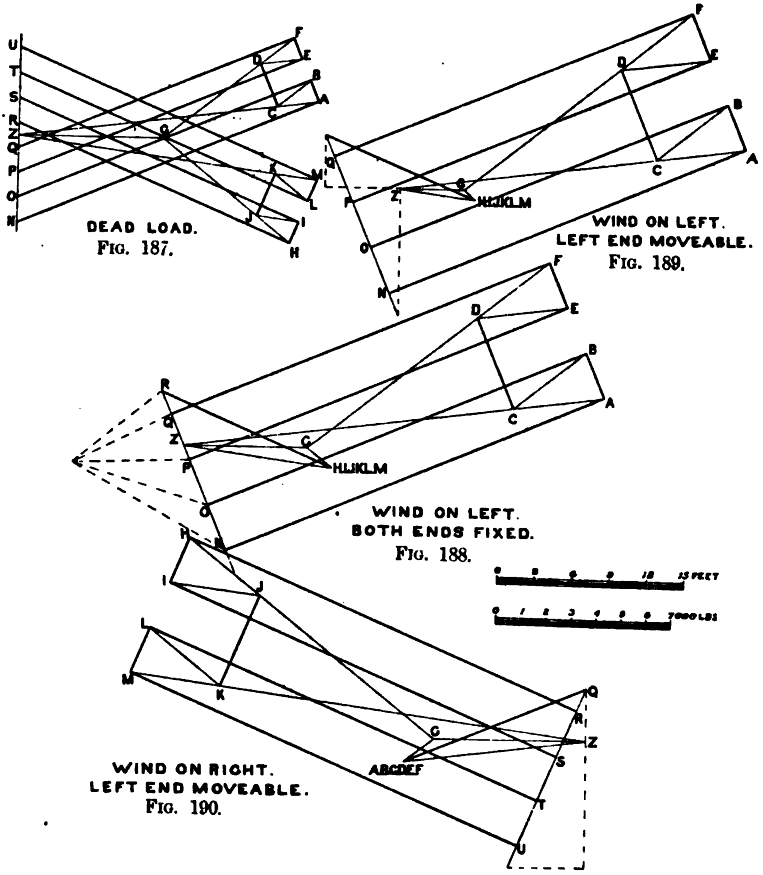


Fig. 186.

Fig. 191 represents a diagram of a polygonal roof-truss, for which



DEAD LOAD.
Fig. 187.

WIND ON LEFT.
LEFT END MOVEABLE.
Fig. 189.

WIND ON LEFT.
BOTH ENDS FIXED.
Fig. 188.

WIND ON RIGHT.
LEFT END MOVEABLE.
Fig. 190.

reciprocal diagrams have been drawn for the dead load and wind pressures, Figs. 192 to 195.

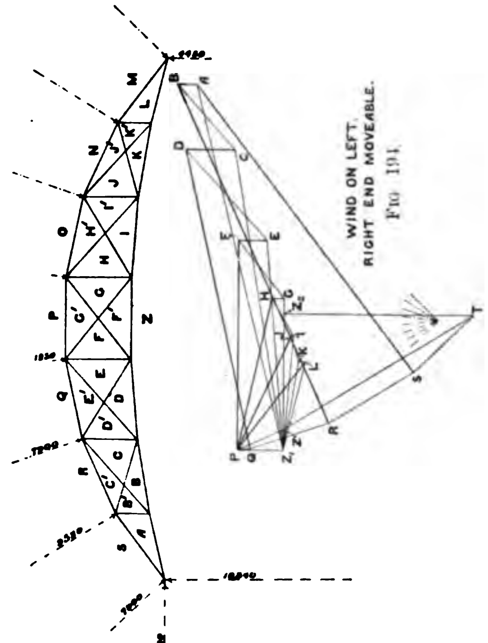
The following data have been employed:—

Span of roof-truss	100·0 ft.
Distance between principals	20·0 „
Dead load per sq. ft. of area covered	14 lbs.
Horizontal wind pressure per sq. ft.	50 „
Dead load at joints 2, 3, 4, 5, 6 and 7	4000 „
„ „ „ 1 and 8	2000 „
Wind pressure at joints 1 and 8	7900 „
„ „ „ 2 „ 7	9520 „
„ „ „ 3 „ 6	7900 „
„ „ „ 4 „ 5	1250 „

The figure is redundant, but one set of diagonals is supposed to be omitted, otherwise the reciprocal stress diagrams could not be drawn.

Fig. 192 represents the stresses due to the dead load. Fig. 193 represents the stresses due to wind acting on the left side when both ends of the principal are fixed in position. The point Z is determined by means of the polar and funicular polygons shown in dotted lines. The force polygon for the loads P_2, P_3, and P_4, which produce reactions at both supports, is shown in Fig. 193, *viz.*, PQRSZP, in which SR, RQ, and PQ are drawn parallel to P_2, P_3, and P_4 (Fig. 191). PS is the resultant, and equals the sum of the reactions at the supports due to the three loads under consideration. The force polygon for the four forces P_1, P_2, P_3, and P_4 is PQRSTZP, in which ST is drawn parallel to P, and ZT joined.

The remainder of Fig. 193 will be readily understood by the student, as Bow's notation has been employed throughout.



WIND ON LEFT.
RIGHT END MOVEABLE.
FIG. 191.

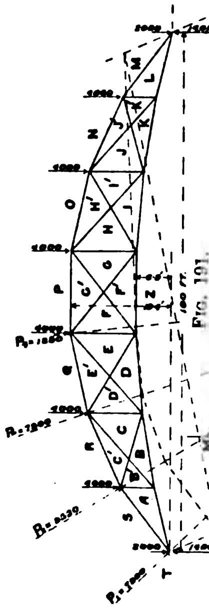
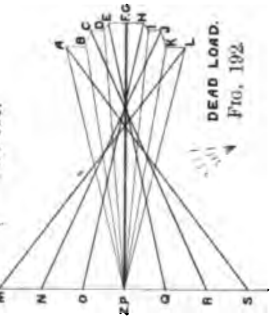


FIG. 191.



DEAD LOAD.
FIG. 192.

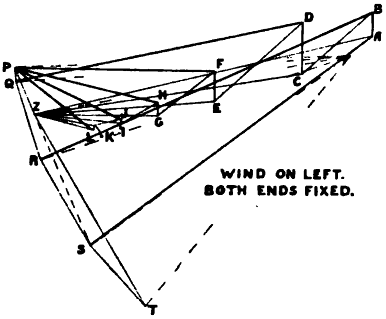


FIG. 193.

Fig. 194 represents the stresses due to wind acting on the left-hand side, but with the left end fixed and the right on the point of moving; under these circumstances the left support must supply the horizontal component of the inclined reactions at both supports. The point Z in Fig. 193 becomes Z_1 in Fig. 194; PZ_1 represents the vertical reaction at the right support, and TZ_2 represents the vertical reaction at the left support.

The horizontal reaction at the left support is represented by Z_1 Z_2; the stresses in the various members of the truss are determined by completing the figure as before.

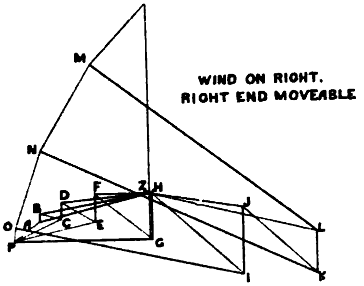


FIG. 195.

Fig. 195 represents the stresses due to wind acting on the right with the right end movable as before.

The stresses due to the various cases of loading obtained by measuring the reciprocal diagrams, Figs. 192 to 195, and also the total maxima stresses, are given in the following table :—

TABLE XLI.

Bar.	Dead load stresses.	Both ends of principal fixed to the support.		Right end movable, left end fixed.		Total maxima stresses.
		Wind on the right.	Wind on the left.	Wind on the right.	Wind on the left.	
	lbs.	lbs.	lbs.	lbs.	lbs.	lbs.
SA	- 29,200	- 9,780	- 33,910	+ 3,110	- 34,010	- 63,210
RC′	- 27,460	- 11,510	- 35,130	+ 5,000	- 35,680	- 63,140
QE′	- 26,240	- 14,230	- 28,350	+ 8,000	- 28,680	- 54,920
PG′	- 25,570	- 19,340	- 19,340	- 13,340	- 20,180	- 45,750
OH′	- 26,240	- 28,350	- 14,230	- 22,460	- 14,900	- 54,590
NJ′	- 27,460	- 35,130	- 11,510	- 29,130	- 12,170	- 62,590
ML	- 29,200	- 33,910	- 9,780	- 27,120	+ 10,780	- 63,110
AZ	+ 24,120	+ 5,890	+ 33,570	- 10,900	+ 35,350	+ 59,470
BZ	+ 23,570	+ 5,780	+ 26,180	- 10,670	+ 29,640	+ 53,210
DZ	+ 24,960	+ 8,450	+ 17,450	- 8,450	+ 20,230	+ 45,190
F′Z	+ 25,570	+ 11,900	+ 11,900	- 5,220	+ 14,560	+ 40,130
IZ	+ 24,960	+ 17,450	+ 8,450	+ 230	+ 11,010	+ 42,410
KZ	+ 23,570	+ 26,180	+ 5,780	+ 9,120	+ 8,670	+ 49,750
LZ	+ 24,120	+ 33,570	+ 5,890	+ 16,450	- 8,840	+ 57,690
AB′	+ 1,950	+ 500	- 2,450	- 780	- 1,890	+ 2,450 - 500
CD′	+ 1,220	- 450	- 5,670	- 1,670	- 4,780	+ 1,220 - 4,450
EF	+ 3,390	- 1,340	- 2,890	- 2,670	- 2,780	+ 3,390
GH	+ 3,390	- 2,890	- 1,340	- 4,340	- 1,220	+ 3,390 - 950
I′J	+ 1,220	- 5,670	- 450	- 6,450	- 230	+ 1,220 - 5,230
K′L	+ 1,950	- 2,450	+ 500	- 4,000	+ 670	+ 2,620 - 2,050
BB′	0	0	+ 9,120	0	+ 8,340	+ 9,120
DD′	0	0	+ 11,450	0	+ 11,670	+ 11,670
FF′	0	0	+ 7,000	0	+ 7,110	+ 7,110
HI	+ 890	0	+ 4,000	0	+ 4,110	+ 5,000
JK	+ 1,670	0	+ 2,780	0	+ 2,400	+ 4,450
BC	+ 1,670	+ 2,780	0	+ 2,170	0	+ 4,450
DE	+ 890	+ 4,000	0	+ 3,780	0	+ 4,890
F′G	0	+ 7,000	0	+ 7,000	0	+ 7,000
II′	0	+ 11,450	0	+ 11,780	0	+ 11,780
KK′	0	+ 9,120	0	+ 9,560	0	+ 9,560

CHAPTER VIII.

BRACED GIRDERS WITH PARALLEL FLANGES.

FIG. 196 represents a cantilever loaded at the apices. The load $\frac{W}{2}$ at the apex 0 and the stresses in the bars 01 and 02 are in equilibrium; they may therefore be represented by the triangle of forces *abc*, in which *ac* represents the load $\frac{W}{2}$, Fig. 197.

The stress in 01 (01_1), may be found thus:—

$$01_1 : \frac{W}{2} : : ab : ac \qquad \therefore 01_1 = \frac{W}{2} \sec \theta$$

$$\text{also } 02_1 : \frac{W}{2} : : bc : ac \qquad \therefore 02_1 = \frac{W}{2} \tan \theta.$$

We see also, from the direction of the forces in the triangle *abc*, that 02 is in tension, and 01 is in compression.

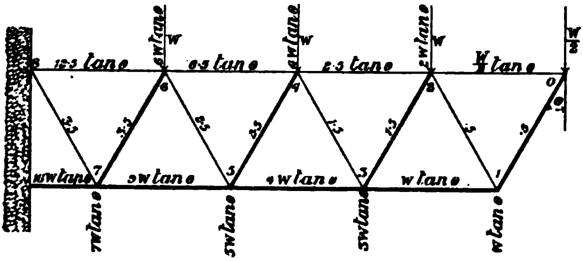


FIG. 196.

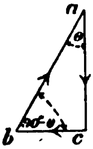


FIG. 197.

The stress in 1 2 is clearly the same as in 01. The stress in 2 3 due to the load at the apex 2 is W sec θ, and the total stress on 2 3 is therefore $\frac{3W}{2}$ sec θ, which is also the stress in the bar 3 4; the former is in compression, and the latter in tension.

The stress in 1 3 is the sum of the horizontal components of the stresses in 01 and 1 2.

$$\therefore 1\ 3_1 = W \tan \theta \text{ (compression).}$$

The stress $2\,4_1$ is the sum of the horizontal components of the stresses in 1 2 and 2 3, plus the stress in 02.

$$\therefore 2\,4_1 = 2\mathrm{W}\tan\theta + \frac{\mathrm{W}}{2}\tan\theta \text{ (tension).}$$

In a similar manner we may find the stresses in the remaining bars.

We see that the increment of stress in the web is $\mathrm{W}\sec\theta$, so that we may write down the stresses in the remaining bars thus:—

In the bars 4 5 and 5 6 the stress is $\frac{5\mathrm{W}}{2}\sec\theta$.
,, ,, 6 7 ,, 7 8 ,, ,, $\frac{7\mathrm{W}}{2}\sec\theta$.

The increment of flange stress at the apex 1 we found to be $\mathrm{W}\tan\theta$, and at the apex 2, $2\mathrm{W}\tan\theta$. In a similar manner we could have shown that the increment of flange stress at the apex 3 is $3\mathrm{W}\tan\theta$, and at the apex 4 it is $4\mathrm{W}\tan\theta$, and so on for the remaining bars.

Let W = 10 tons, and $\theta = 30°$,

$$\text{then } \sec\theta = 1{\cdot}154, \text{ and } \tan\theta = 0{\cdot}577.$$

The increment of web stress $\mathrm{W}\sec\theta = 11{\cdot}54$ tons, and $\mathrm{W}\tan\theta = 5{\cdot}8$ tons.

In Fig. 196 we may write down the stresses in the diagonals by multiplying the number written on the bars by 11·54, thus:—

Stress in 0 1 = $0{\cdot}5 \times \mathrm{W}\sec\theta = -\ 5{\cdot}77$ tons, compression
,, 1 2 = $0{\cdot}5 \times \mathrm{W}\sec\theta = +\ 5{\cdot}77$,, tension
,, 2 3 = $1{\cdot}5 \times \mathrm{W}\sec\theta = -\ 17{\cdot}31$,, compression
,, 3 4 = $1{\cdot}5 \times \mathrm{W}\sec\theta = +\ 17{\cdot}31$,, tension
,, 4 5 = $2{\cdot}5 \times \mathrm{W}\sec\theta = -\ 28{\cdot}85$,, compression
,, 5 6 = $2{\cdot}5 \times \mathrm{W}\sec\theta = +\ 28{\cdot}85$,, tension
,, 6 7 = $3{\cdot}5 \times \mathrm{W}\sec\theta = -\ 40{\cdot}39$,, compression
,, 7 8 = $3{\cdot}5 \times \mathrm{W}\sec\theta = +\ 40{\cdot}39$,, tension

The stresses in the flanges are written down in a similar manner, thus:—

Stress in 0 2 = + 2·89 tons.
,, 2 4 = + 14·45 ,,
,, 4 6 = + 37·58 ,,
,, 6 8 = + 72·25 ,,

Stress in 23 = − 5·77 tons
,, 35 = − 23·08 ,,
,, 57 = − 51·93 ,,
,, 79 = − 92·32 ,,

Fig. 198 represents a triangular girder supported at both ends and loaded symmetrically; such a girder is termed a "Warren girder". It would be used for a deck bridge, the deck being carried by the top member.

The reactions are each equal to half the load. The stresses in the bars are found thus:—

Stress in $\mathrm{KS} = \mathrm{R}_1 \sec\theta \quad = +2{\cdot}5\mathrm{W}\sec\theta$ (tension)
„ $\mathrm{KJ} = \mathrm{R}_1.\sec\theta \quad = -2{\cdot}5\mathrm{W}\sec\theta$ (compression)
„ $\mathrm{IJ} = (\mathrm{R}_1 - \mathrm{W})\sec\theta \quad = +1{\cdot}5\mathrm{W}\sec\theta$
„ $\mathrm{IH} = (\mathrm{R}_1 - \mathrm{W})\sec\theta \quad = -1{\cdot}5\mathrm{W}\sec\theta$
„ $\mathrm{HG} = (\mathrm{R}_1 - \mathrm{W} - \mathrm{W})\sec\theta = +0{\cdot}5\mathrm{W}\sec\theta$
„ $\mathrm{GF} = (\mathrm{R}_1 - \mathrm{W} - \mathrm{W})\sec\theta = -0{\cdot}5\mathrm{W}\sec\theta$

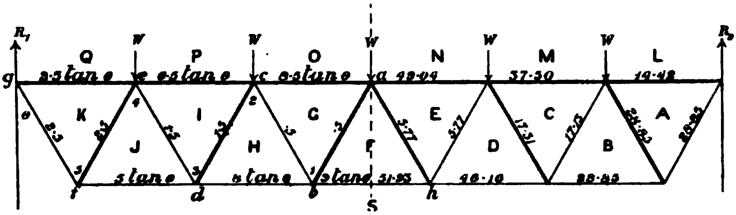


FIG. 198.

Hence we may write coefficients against the bars, and multiply by W sec θ to find the stress as before. It should be noted that the bars which slope towards the centre are in compression, while those which slope away from the centre are in tension.

The stresses in the flanges are found by adding the horizontal components of the stresses in the web, as before, thus:—

Stress in $\mathrm{QK} = \quad = -2{\cdot}5\mathrm{W}\tan\theta$
„ $\mathrm{PI} = -(2{\cdot}5+4)\mathrm{W}\tan\theta = -6{\cdot}5\mathrm{W}\tan\theta$
„ $\mathrm{OG} = -(6{\cdot}5+2)\mathrm{W}\tan\theta = -8{\cdot}5\mathrm{W}\tan\theta$
„ $\mathrm{JS} = \quad = +5{\cdot}0\mathrm{W}\tan\theta$
„ $\mathrm{HS} = +(5+3)\mathrm{W}\tan\theta \quad = +8{\cdot}0\mathrm{W}\tan\theta$
„ $\mathrm{FS} = +(8+1)\mathrm{W}\tan\theta \quad = +9{\cdot}0\mathrm{W}\tan\theta$

Here we note that by adding the coefficients on two bars which meet at a vertex, and writing the result on the vertex, we have the coefficient for the increment of flange stress, which requires to be multiplied by W tan θ and added to the stress in the flange preceding it, counting from the left support, to find the stress.

Let $\theta = 30°$, and $\mathrm{W} = 10$ tons.

$$\text{Then } \sec\theta = 1{\cdot}154 \quad \therefore\ \mathrm{W}\sec\theta = 11{\cdot}54 \text{ tons,}$$
$$\tan\theta = 0{\cdot}577 \quad \therefore\ \mathrm{W}\tan\theta = 5{\cdot}77 \text{ tons.}$$

The actual stresses are written on the right-hand half of the girder, Fig. 198.

Fig. 199 shows a lattice girder loaded symmetrically on the top and bottom flanges.

Here we may consider each triangulation separately, and apply the foregoing principles. The results are written on the left-hand half of the girder, Fig. 199.

Let $W_1 = 5$ tons, and $W = 10$ tons, $\theta = 45°$.[1]

Then $\sec\theta = 1{\cdot}414$

$\therefore W_1 \sec\theta = 70{\cdot}7$; $W \sec\theta = 14{\cdot}14$; $\tan\theta = 1$.

The results are written on the right-hand half of the girder, Fig. 199.

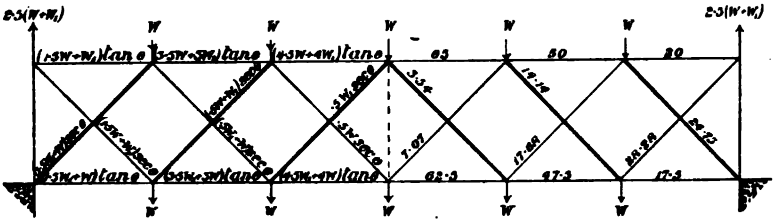
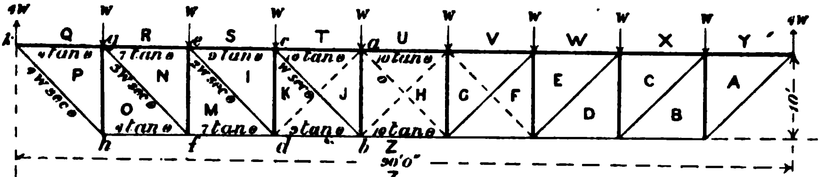


FIG. 199.

Fig. 200 represents a truss known as the Murphy-Whipple, or Pratt, truss. The vertical members are constructed as compression members, while the diagonal members are in tension. The truss as

FIG. 200.



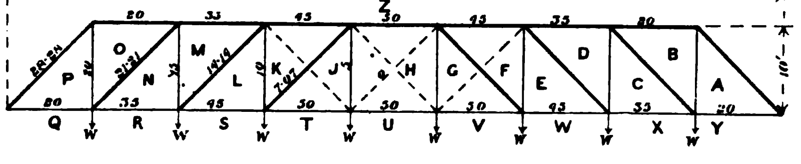


FIG. 201.

shown would be used as a deck bridge like the Warren girder, Fig. 198. Fig. 201 represents the same truss inverted, but the vertical members are in tension, and the diagonal members are in compression. In this form it is termed the Howe truss, and is much used for composite structures, the vertical members consisting of one or more bolts, the diagonal and top members of timber, while the bottom member may be constructed of timber or iron. The Howe truss, as shown in Fig. 201, would be used for a through bridge, and the deck would be carried by the bottom member. In both Figs. 200 and 201, the diagonals in the centre bay are unstressed, with the loads symmetrically arranged on either side of the centre, as shown; they will be referred to in regard to the live load. We may write coefficients against the bars and determine the stresses as before; thus in Fig. 200 the stresses in the verticals are—

---

[1] In the figure W_1 is the load at the top panel points.

$$ab_1 = \text{W},\ cd_1 = 2\text{W},\ ef_1 = 3\text{W},\ gh_1 = 4\text{W}.$$

The stresses in the diagonals are—

$$cb_1 = \text{W} \sec \theta,\ de_1 = 2\text{W} \sec \theta, fg_1 = 3\text{W} \sec \theta,\ hk_1 = 4\text{W} \sec \theta.$$

The horizontal components of the stresses in the diagonals will produce stresses in the top and bottom members, thus:—

$$\begin{aligned} kg_1 &= 4 \tan \theta \\ ge_1 &= 4 \tan \theta + 3 \tan \theta = 7 \tan \theta \\ ec_1 &= 7 \tan \theta + 2 \tan \theta = 9 \tan \theta \\ ca_1 &= 9 \tan \theta + \tan \theta = 10 \tan \theta. \end{aligned}$$

In a similar manner we may find the stresses in the bottom member; they are written in Fig. 200.

$$\text{If W} = 5 \text{ tons, and } \theta = 45^\circ$$
$$\text{then W} \sec \theta = 5 \times 1{\cdot}414 = 7{\cdot}07 \text{ tons; W} \tan \theta = 5 \text{ tons.}$$

The results are written on Fig. 201.

We may draw reciprocal figures to find the stresses in the various members as illustrated in Figs. 202 and 203.

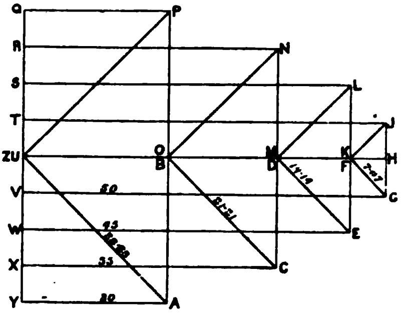


FIG. 202.

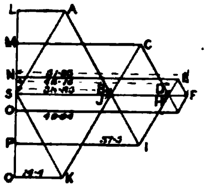


FIG. 203.

Fig. 202 represents the stresses in the truss illustrated in Fig. 201, and Fig. 203 the stresses in that illustrated in Fig. 198.

Braced Girders with Parallel Flanges subjected to a Travelling Load.—We have seen in Chapter IV., Fig. 89, that the maximum shearing stress at any point of a girder subjected to a travelling load occurs when the load extends from either abutment to the point in question, the remainder of the girder being unloaded. Hence in the Warren girder, Fig. 198, the maximum compressive stress in the bar FG occurs when the apices on the right half of the girder are loaded. The maximum tensile stress in FG occurs when the remaining apices are loaded, and the apices on the right half of the girder are unloaded, thus:—

$$-\text{FG} = \frac{\text{W}\sec\theta}{6}(1 + 2 + 3) = -\text{W}\sec\theta$$

$$+\text{FG} = \frac{\text{W}\sec\theta}{6}(1 + 2) = +\frac{\text{W}\sec\theta}{2}$$

The same stresses will be produced on the bar GH, only with the signs changed, thus:—

$$-\text{GH} = -\frac{\text{W}\sec\theta}{2}, \text{ and } +\text{GH} = +\text{W}\sec\theta.$$

In a similar manner the stresses in all the inclined bars may be written down; thus for the left half of the girder we have—

$$\text{Stress in HI} = -\frac{\text{W}\sec\theta}{6}(1 + 2 + 3 + 4) \text{ and } +\frac{\text{W}\sec\theta}{6}(1)$$

$$= -\frac{10\text{W}\sec\theta}{6} \text{ and } +\frac{\text{W}\sec\theta}{6}$$

$$\text{,, ,, IJ} = +\frac{10\text{W}\sec\theta}{6} \text{ and } -\frac{\text{W}\sec\theta}{6}$$

$$\text{,, ,, JK} = -\frac{\text{W}\sec\theta}{6}(1 + 2 + 3 + 4 + 5) = -\frac{15\text{W}\sec\theta}{6}$$

$$\text{,, ,, KS} = +\frac{\text{W}\sec\theta}{6}(1 + 2 + 3 + 4 + 5) = +\frac{15\text{W}\sec\theta}{6}$$

The maximum stresses in the horizontal members due to the live load will occur when the girder is fully loaded, and they may be found in a similar manner to that explained in the foregoing examples. We may determine the stresses in the inclined members graphically thus: In the Warren girder, Fig. 198, let the dead load at the apices be 5 tons, and the live load 10 tons. Here we draw the ordinary shearing-stress diagrams for the live and dead loads, and set out the triangles as shown in Fig. 204.

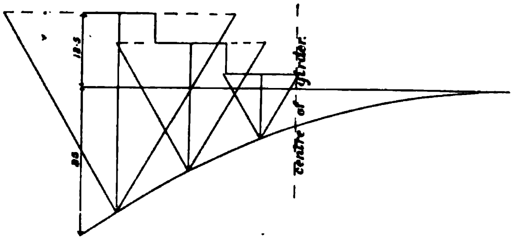


FIG. 204.

In Fig. 200 let the dead load at the apices be 5 tons, and the live load 10 tons, and let $\theta = 45°$. The stresses in the top and bottom members may be found as in the case of the dead load, or we may use the method of sections; thus for the stress in WE, we suppose a vertical section cutting ED, and take moments about the point where ED cuts the bottom member, thus:—

$$-10WE + 15(20 + 10) - 60 \times 30 = 0$$
$$\therefore WE = -135 \text{ tons.}$$

In a similar manner we write down the stresses in all the top and bottom members, thus:—

$$YA = \qquad = -60 \text{ tons}$$
$$XC = \frac{-60 \times 20 + 15 \times 10}{10} = -105 \text{ ,,}$$
$$WE = \frac{-60 \times 30 + 15(20 + 10)}{10} = -135 \text{ ,,}$$
$$VG = \frac{-60 \times 40 + 15(30 + 20 + 10)}{10} = -150 \text{ ,,}$$
$$UH = \qquad = -150 \text{ ,,}$$
$$BZ = \qquad = +60 \text{ ,,}$$
$$DZ = \frac{60 \times 20 - 15 \times 10}{10} = +105 \text{ ,,}$$
$$FZ = \frac{60 \times 30 - 15(20 + 10)}{10} = +135 \text{ ,,}$$
$$HZ = \frac{60 \times 40 - 15(30 + 20 + 10)}{10} = +150 \text{ ,,}$$

These stresses are three times those written on Fig. 201.

Stresses in Diagonal Members.—For the stresses in the diagonal members we proceed as in the case of the Warren girder, Fig. 198:—

$$W_1 \sec\theta = 14{\cdot}14 \text{ tons for the live load}$$
$$W \sec\theta = 7{\cdot}07 \text{ ,, ,, dead load}$$
$$\text{and } \frac{14{\cdot}14}{9} = 1{\cdot}572.$$

The stresses on NO, for example, may be found most conveniently by writing the equation so that the live loads producing tension and compression may be readily seen:—

$$NO = \left.\begin{matrix}21{\cdot}21 + 1{\cdot}572(1 + \ldots + 7) \\ -1{\cdot}572(1)\end{matrix}\right\} = \left.\begin{matrix}21{\cdot}21 + 44{\cdot}016 \\ -1{\cdot}57\end{matrix}\right\}$$

hence NO = + 65·22 tons and + 19·64 tons.

The stress in the bar PZ is—

$$PZ = 28{\cdot}28 + 1{\cdot}572(1 + \ldots + 8) = +84{\cdot}84 \text{ tons.}$$

Also in the remaining bars we may find the stresses as before:—

$$\mathrm{IM} = 14{\cdot}14 + \left.\begin{array}{l} 1{\cdot}572(1 + \ldots + 6) \\ - 1{\cdot}572(1 + 2) \end{array}\right\}$$

$$\therefore \mathrm{IM} = 14{\cdot}14 + \left.\begin{array}{l} 33{\cdot}012 \\ - 4{\cdot}716 \end{array}\right\}$$

$$\therefore \mathrm{IM} = + 47{\cdot}15 \text{ tons and } + 9{\cdot}42 \text{ tons.}$$

The stresses in the first three bars are not reversed by the live load, and are always tensile. In the bar JK we shall see that the live load puts a compressive stress upon the bar, thus:—

$$\mathrm{JK} = 7{\cdot}07 + \left.\begin{array}{l} 1{\cdot}572(1 + \ldots + 5) \\ - 1{\cdot}572(1 + 2 + 3) \end{array}\right\}$$

$$\therefore \mathrm{JK} = 7{\cdot}07 + \left.\begin{array}{l} 23{\cdot}58 \\ - 9{\cdot}432 \end{array}\right\}$$

$$\therefore \mathrm{JK} = + 30{\cdot}65 \text{ tons and } - 2{\cdot}36 \text{ tons.}$$

Since JK can only resist tensile stresses, we must counterbrace the panel by inserting a diagonal crossing JK. Since it is hardly possible to ensure that each member shall perform the duty assigned to it, it is better practice to make the counterbrace take all the live-load shear, so that JK would have to sustain + 30·65, and the counterbrace 9·432 tons.

In the central bay there is no stress in the diagonals due to the dead load; the stresses due to the live load are—

$$\pm\, 1{\cdot}572(1 + 2 + 3 + 4) = \pm\, 15{\cdot}72 \text{ tons.}$$

Hence the central bay must be counterbraced by inserting two diagonals crossing each other, each designed for a tensile stress of 15·72 tons. These diagonal counterbraces are shown in dotted lines, both in Figs. 200 and 201. The counterbraces only come into action when the live load tends to produce distortion of the panel by putting a compressive stress on the main brace; when this occurs, the main brace bends and throws the tensile stress upon the counterbrace, so that the two braces are never in action simultaneously.

Stresses in Vertical Members.—The stresses in the verticals are the vertical components of the stresses in the diagonals, and one may be found from the other, or we may proceed independently, thus:—

$$\mathrm{IK} = - 10 \left.\begin{array}{l} - \tfrac{10}{9}(1 + \ldots + 5) \\ + \tfrac{10}{9}(1 + 2) \end{array}\right\} \therefore \mathrm{IK} = - 26{\cdot}6 \text{ tons and } - 6{\cdot}66$$

$$\mathrm{JH} = - 5 \left.\begin{array}{l} - \tfrac{10}{9}(1 + \ldots + 4) \\ + \tfrac{10}{9}(1 + 2 + 3) \end{array}\right\} \therefore \mathrm{JH} = - 16{\cdot}1 \text{ and } + 1{\cdot}66.$$

Hence we see that the live load puts tensile stress upon JH and HG. The maximum stresses are written on Fig. 205.

In designing these girders we must compare the maximum stresses, as written on Fig. 205, with those produced by the dead load only, and thus find the range of stress for those bars where the stress is not reversed by the live load. Where there is a reversal of stress, as in the two central diagonals, the range of stress is shown on Fig. 205. Or the equivalent dead load may be determined and the intensity of working stress found as explained in Chapter I.

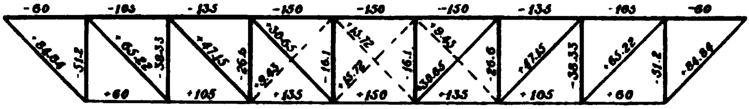


FIG. 205.

When there are more than one system of triangulation each system may be calculated separately, as in Fig. 206, as the loads upon the apices of a particular system are transmitted by that system to the abutment independently of the others.

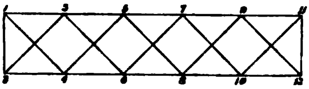


FIG. 206.

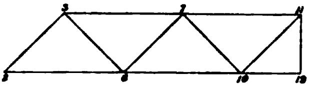


FIG. 207.

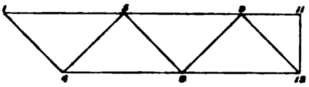


FIG. 208.

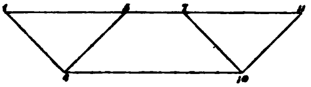


FIG. 209.

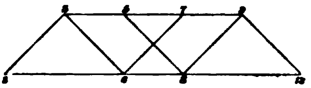


FIG. 210.

Fig. 206 represents a lattice girder or truss of an odd number of panels, which may be subdivided into four different systems, as shown in Figs. 207 to 210. The subdivision illustrated in Figs. 207 and 208 represents the method for partial loading, and Figs. 209 and 210 show a possible division for complete loading; but here there is an ambiguity, as we are not sure whether the system illustrated in Figs. 207 and 208 would not hold. If the stresses are calculated for each case, and the maximum stresses adopted in the design of the girder, this ambiguity will not cause a want of strength. The student may work out the stresses in this case by assuming a live load of 10 tons and a dead load of 5 tons at each bottom apex, $\theta = 30°$, span $= 50$ ft.

This method of subdivision and superposition is often very convenient, and may be used in the two following examples, Figs. 211 and 214. Here, however, there is no ambiguity, as the number of panels is even, and there are only two subdivisions possible.

Figs. 211 and 214 are both modifications of the Pratt or Murphy-Whipple truss, but in these forms they are sometimes called Linville trusses; the former is suitable for a deck bridge, the latter for a through bridge.

Fig. 214 represents the outline of the bridge over the Ohio River, 415 ft. span.

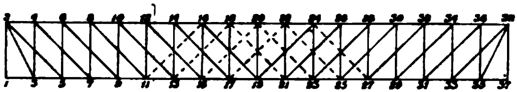


Fig. 211.

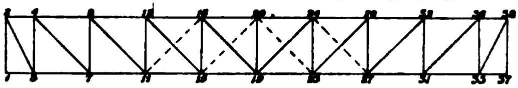


Fig. 212.

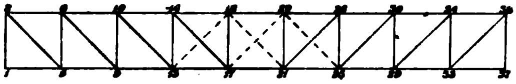


Fig. 213.

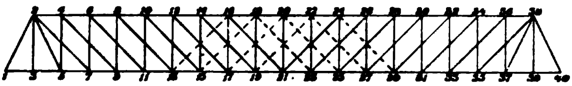


Fig. 214.

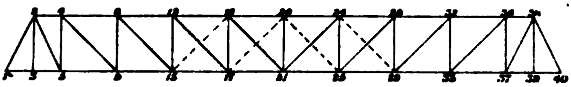


Fig. 215.

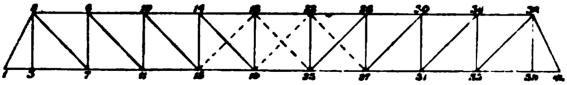


Fig. 216.

Fig. 217 shows a form of timber truss which has been largely used for road bridges in Australia. The vertical tension members are constructed with two or more bolts, which may be screwed up as required, and the diagonal members of the web fit into iron pockets, and are provided with double wedges at the ends for putting the member in initial compression and taking up shrinkage.

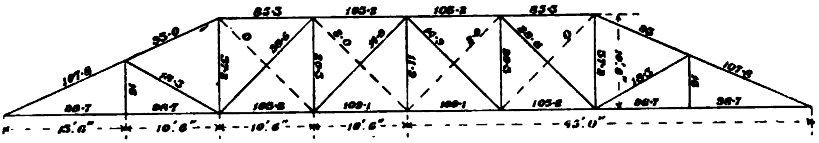


Fig. 217.

The stresses written on the various members of the truss have been determined in the manner sufficiently explained, and may be verified by the student. The dead load is 0·6 ton per foot run, and the live load 0·53 ton per foot run on each truss.

The counterbraces in the four central bays, shown in dotted lines, are chiefly necessary in consequence of the wedges, and the stresses, in the various members of the truss may vary from those written on Fig. 217, depending on tightness or slackness of the wedges. The counterbraces should be made about half the sectional area of the main diagonal in the same bay, and should be arranged with their largest lateral dimensions at right angles to the plane of the truss in order to give lateral stiffness.

The cross-beams in this truss are spaced on each side of the panel points; the main tie is subjected to a certain amount of transverse stress.

Fig. 218 shows a Howe truss as used in a highway bridge of 70 ft. span, in which the vertical members consist of two or more bolts, but the diagonal members of the web and the ends are inclined at the same angle, *viz.* 45°. The cross-beams carrying the floor rest upon the bottom chord at the panel-points, the vertical bolts being spaced longitudinally at a distance apart equal to the width of the floor beam. There are no wedges at the ends of the diagonal members, as the shrinkage of the timber may be taken up completely by screwing up the nuts on the vertical bolts, hence only the central bay is counterbraced.

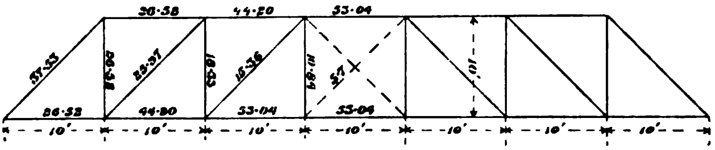


FIG. 218.

The stresses written on Fig. 218 have been determined for a dead load of 0·414 ton and a live load of 0·470 ton per foot run for each truss.

This truss is obviously superior to the foregoing for timber bridges.

The method of constructing the joints in the bottom chord of these trusses will be considered in Chapter XIII.

The bottom chord in both these trusses may be constructed of steel, and the compression members only of timber. This composite truss has the advantage of allowing the various timber members to be removed without stopping the traffic over the bridge.

The following example is given to illustrate the method of calculating the stresses in lattice girders subjected to a permanent load due to the weight of the structure, and a live load due to an engine and train. This girder is not given as an example of good proportion, as it is too deep, and is unsuitable for so small a span.

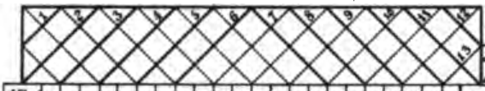
Let the permanent load considered as concentrated at each apex be 2 tons, denoted by W_1, and the live load concentrated in a similar manner be 5 tons, denoted by W. There are 12 bays and 4 systems of triangulation, the bars being inclined at 45°.

$$\text{W} \sec \theta = 5 \times 1{\cdot}414 = 7{\cdot}070 \text{ tons, and } \text{W}_1 \sec \theta = 2{\cdot}828 \text{ tons.}$$

$$\frac{\text{W} \sec \theta}{12} = 0{\cdot}589 \text{ tons.}$$

Span of girder = 60 ft., depth = 10 ft.

By referring to Fig. 219 and the subjoined table of stresses, it will be observed that the live load W_1 produces stresses on bars 3, 7, and 11; that W_2 produces stresses on bars 4, 8, and 12. W_3 also produces stresses on bars 1, 5, 9, and 13, thus:—



Nº of Bar	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	W_9	W_{10}	W_{11}	Stress from Live Load		Stress from Dead Load		Total Stresses from Live and Dead Loads	
												Compⁿ −	Tensⁿ +	Compⁿ −	Tensⁿ +	Compⁿ −	Tensⁿ +
1			-5·30				-2·94				-·59	8·83		2·83		11·76	
2				-4·71				-2·36				7·07		2·83		9·90	
3	+·59				-4·12				-1·77			5·89	·59	2·83		8·72	·59
4		+1·18				-3·53				-1·18		4·71	1·18	1·41		6·12	1·18
5			+1·77				-2·94				-·59	3·53	1·77	0		3·53	1·77
6				+2·36				-2·36				2·36	2·36	0		2·36	2·36
7	+·59				+2·94				-1·77			1·77	3·53	0		1·77	3·53
8		+1·18				+3·53				-1·18		1·18	4·71		1·41	1·18	6·12
9			+1·77				+4·12				-·59	·59	5·89		2·83	·59	8·72
10				+2·36				+4·71					7·07		2·83		9·90
11	+·59				+2·94				+5·30				8·83		2·83		11·76
12		+1·18				+3·53				+5·89			10·60		4·24		14·84
13			+1·77				+4·12				+6·48		12·37		5·66		18·03

FIG. 219.

$$-\frac{\text{W} \sec \theta}{12} = -\ 0{\cdot}589, \text{ say } -\ 0{\cdot}59.$$

W_3 produces—

$$-\frac{9\text{W} \sec \theta}{12} = -\ 5{\cdot}30 \text{ on bar 1.}$$

$$+\frac{3\text{W} \sec \theta}{12} = +\ 1{\cdot}77 \text{ on bars 5, 9, and 13.}$$

The stresses produced on each bar by the various loads, considered separately, are tabulated as shown. The stresses on bar 1 for the loads W_3, W_7 and W_{11} which affect it are added together and the result written in the first column, as it is compressive. In a similar manner, the stresses on the other bars are added, and written in the first or second column, according as the stress is compressive or tensile.

The stresses in the third and fourth columns due to dead load are found, as in the foregoing examples, by affixing the proper coefficient to the bar and multiplying it by $W_1 \sec \theta$ (Fig. 220).

The total maximum stresses are found by adding the stresses of like kind in the first four columns due to live load and dead load. It should be noted that bars from 3 to 9 are subjected to both tensile and compressive stresses.

The stresses in the remaining 13 bars may be written down by numbering them from the right-hand support.

The stresses in the top and bottom horizontal members are found as before, by adding the coefficients written in the lattice bars, Fig. 220, and multiplying by $(W + W_1) \tan \theta$ for the increment of flange stress, $= (2 + 5) \tan \theta = 7 \tan \theta$.

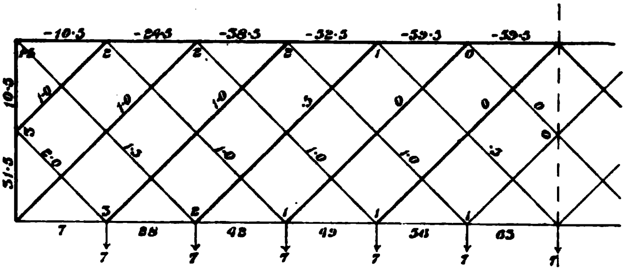


FIG. 220.

Thus the stresses in the top member may be writtten down—

Stress on 1st bay $= 1·5 \times 7 \quad = -10·5$ tons.
,, ,, 2nd ,, $= 10·5 + 2 \times 7 = -24·5$,,
,, ,, 3rd ,, $= 24·5 + 2 \times 7 = -38·5$,,

The stresses in both top and bottom members are written down in Fig. 220.

The stresses in the top and bottom members may be plotted to scale as ordinates on the length of the girder as a base, and the extremities joined, forming two polygons, which are the diagrams of direct stresses due to bending, as explained in Chapter IV., and may be used for designing the top and bottom flanges.

Some Assumptions made in Calculating Stresses.—In the examples illustrated in Figs. 219 and 220 each system of triangulation of the web has been assumed to act independently just as if the remaining systems were not there. This is not strictly true as the upper and lower members consist of riveted sections built up to form one piece without hinges, and each system must be more or less affected by the loads and stresses in the remaining systems. The assumption made is, however, as good as any other, and leads to results which are sufficiently accurate under the circumstances. Again in the Pratt truss, and others, the dead load is assumed to be concentrated at the bottom joints in a through bridge, and at the top joints in a deck bridge, whereas a portion of the dead load must be carried by both the upper and lower joints.

If the dead load acting at the panel-points is made up as follows:—

Deck	=	8,000 pounds
Floor girders	=	10,000 ,,
Main trusses	=	22,000 ,,
Total	=	40,000 ,,

For one girder the total dead load at the panel-points is 20,000 pounds, and it would be more accurate to take the load acting at the upper panel-points in a through bridge as half the panel weight of the main truss, or 5500 pounds, and the remaining 14,500 pounds as acting at the bottom panel-points. This division of the load between the upper and lower joints in ordinary trusses, such as those already dealt with, does not affect the total stresses to an important extent, but in large spans it is much more important to divide the dead load, as it bears a relatively greater ratio to the total load than in the smaller bridges.

An example will now be given of a Pratt truss of 150 ft. span, and 28 ft. depth consisting of six panels each 25 ft. long suitable for a through bridge supporting one line of railway carrying two locomotives and train load producing panel-point pressures of 80,000 pounds on each truss. This train load is approximately equivalent to Mr. Cooper's[1] Class E_{50}. The dead load panel-point pressure was known to be 25,000 pounds; about 7000 pounds would act at the top panel-points, and 18,000 pounds at the bottom panel-points. The panel-point loads may be conveniently expressed in units of 1000 pounds giving 7, 18 and 80 for the top and bottom dead loads and the bottom live load respectively. Fig. 221 represents the truss with the dead and live load stresses written on the various members denoted by the letters *d* and *l* respectively. It will not be necessary to illustrate the method of determining the dead load stresses as they may be easily verified by the student. Sec $\theta = 1{\cdot}341$, tan $\theta = 0{\cdot}893$.

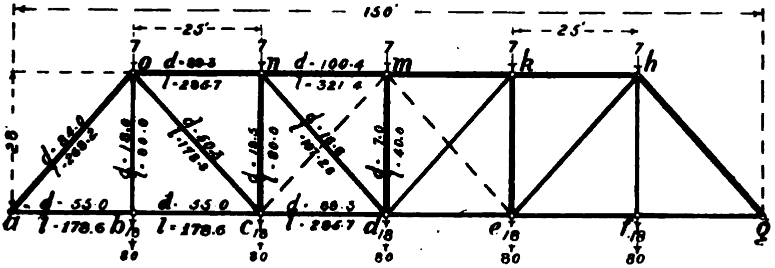


FIG. 221.

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[1] *Specification of Railroad Bridges*, 1901.

Live Load Stresses.—The live load stresses may be determined in the following manner:—

Member *md:* The maximum stress occurs in *md* when only the panel-points *f* and *e* are loaded with 80 units—

$$\frac{80}{6}(1 + 2) = 40 \text{ units.}$$

Member *nd:* The maxima stresses occur in *nd* when either the panel-points *f*, *e* and *d*, or the panel-points *b* and *c*, are loaded. The former produces tension, and the latter compression, in *nd.*

$$\left\{\frac{80}{6}(1 + 2 + 3)\right\} 1{\cdot}341 \text{ and } - \left\{\frac{80}{6}(1+2)\right\} 1{\cdot}341.$$

$$+ 107{\cdot}28 \quad ,, \quad - 53{\cdot}64.$$

The total stress in *nd* is—

$$107{\cdot}28 + 16{\cdot}8 = 124{\cdot}08 \text{ and } 16{\cdot}8 - 53{\cdot}64 = - 36{\cdot}84.$$

Instead of designing the member *nd* to resist a tensile and compressive stress of 124·08 and 36·84 units respectively, we may insert a counterbrace *mc* to resist the whole of the live load stress 53·64 in tension, and it is usual to add about 10 units for initial tension, giving a total stress of 63·64 units. A similar counterbrace will also be inserted between *m* and *e*. The truss is erected with the adjustable counterbraces slack, and afterwards the initial tension is applied by screwing up the union screw buckles.

Member *oc:* The stress in *oc* is found in a similar manner to be—

$$\left\{\frac{80}{6}(1 + 2 + 3 + 4)\right\} 1{\cdot}341 \text{ and } - \frac{80}{6} \times 1{\cdot}341.$$

$$178{\cdot}8 \quad ,, \quad - 17{\cdot}88.$$

The total stress in *oc* is: $178{\cdot}8 + 50{\cdot}3 = 229{\cdot}1$ units, and $50{\cdot}3 - 17{\cdot}88 = 32{\cdot}5$ units. Here there is no reversal of stress due to the live load, but the dead load stress of 50·3 is reduced to 32·5 when only the panel-point *b* is loaded with 80 units.

Member *ob:* The stress in *ob* is due to the reaction of the cross-girder attached to the member *ob* for live and dead load—

$$18 + 80 = 98 \text{ units.}$$

If the reaction exceeds 80, the result must be added to 18 in order to find the total stress.

Member *nc:* The stress in *nc* is the vertical shear in the panel *cd*—

$$- \frac{80}{6}(1 + 2 + 3) \text{ and } + \frac{80}{6}(1 + 2)$$

$$- 80 \text{ and } + 40 \text{ units.}$$

The total stress in *nc* is—

$$- 80 - 19{\cdot}8 = - 99{\cdot}8, \text{ and } + 40 - 19{\cdot}8 = + 20{\cdot}2.$$

This member must therefore be designed for a compressive stress of − 99·8 and a tensile stress of 20·2 units.

Member *ao:* The live load stress in *ao* is equal to the reaction multiplied by sec θ—

$$-200 \times 1{\cdot}341 = -268{\cdot}2 \text{ units.}$$

The total stress $= -(84 + 268{\cdot}2) = -352{\cdot}2$ units.

Member *on:* We may use the method of chord increments or take moments about *c*, thus:—

$$-\frac{200 \times 50 - 80 \times 25}{28} = -285{\cdot}7 \text{ units.}$$

The total stress in *on* $= -(285{\cdot}7 + 89{\cdot}3) = -375$ units. The stress in *cd* is the same as that in *on*, excepting that *cd* is in tension.

Member *nm:* We may take moments about *d*, then—

$$-\left\{\frac{200 \times 75 - 80(25 + 50)}{28}\right\} = -321{\cdot}4 \text{ units.}$$

The total stress $= -321{\cdot}4 - 100{\cdot}4 = 421{\cdot}8$ units.

Position of Uniform Load for the Maximum Stress in the Diagonals.—If the maximum stress in *nd* is required for a uniform live load, it is usual to consider all the joints loaded from the right support *g* to the joints *d* with the full panel-point live load; in this case 80 units. This is on the safe side, but it is not strictly accurate. To produce 80 units at *d*, the live load must extend up to *c*; but this would produce a load of 40 units at *c*, diminishing the stress in *nd*. If the uniform live load extends just up to *d* the panel-point load is only 40 units at *d*. Hence there must be some intermediate point beyond *d* towards *c* at which the load should terminate, in order to produce the maximum stress in *nd*.

Let x denote the distance from *d* to the left of the uniform live load, w tons per foot run $= 3{\cdot}2$.

The reaction at *c* for this position is $\frac{wx^2}{50}$.

„ „ *d* „ „ „ $wx - \frac{wx^2}{50}$.

The portion of the total shearing stress due to the panel load at *d* for a load extending to a distance x from *d* is—

$$\frac{3}{6}\left(wx - \frac{wx^2}{50}\right) - \frac{2}{6}\left(\frac{wx^2}{50}\right) = \frac{w}{6}\left(3x - \frac{5x^2}{50}\right)$$
$$= \frac{w}{6}\left(3x - \frac{x^2}{10}\right).$$

This is a maximum when $3 - \frac{x}{5} = o$, or $x = 15$ ft. The load at *d* is—

$$40 + \frac{15 \times 3{\cdot}2}{2} = 40 + 24 = 64.$$

The value of x beyond any panel-point to the left must be calculated for every panel to obtain the load at the panel-point producing the maximum stress in the diagonal of that panel, if we require the exact stress.

CHAPTER IX.

BOWSTRING AND POLYGONAL GIRDERS.

THE stresses in girders or trusses with curved or polygonal flanges or chords may be most conveniently ascertained by the method of moments explained in Chapter III.

Let Fig. 222 denote an inverted bowstring girder or truss, in which the bottom chord is a polygon inscribed in a parabolic curve. Let the span be 48 ft., the central depth 6 ft., the number of panels 8, the length of each panel 6 ft., the live load equivalent to 5 tons at each panel-point, the dead load 1 ton at each panel-point.[1]

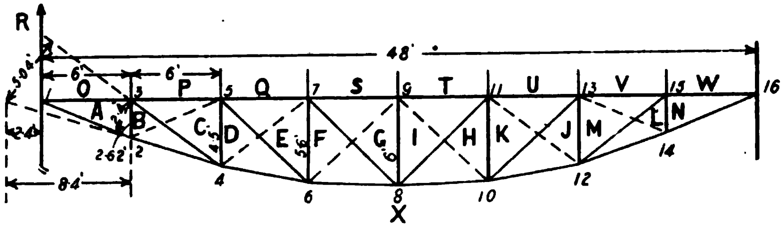


FIG. 222.

The reaction at the left support due to all the panel-points being fully loaded is—

$$R = (\tfrac{1}{8} + \tfrac{2}{8} + \ . \ . \ . \ + \tfrac{7}{8}) + 5(\tfrac{1}{8} + \tfrac{2}{8} + \ . \ . \ . \ + \tfrac{7}{8})$$
$$= 3{\cdot}5 + 17{\cdot}5.$$

The stress in AO, denoted by AO, is found by taking moments about the point 2, the lever arm or the length 32 being 2·62 ft. :—

$$AO(2{\cdot}62) + R(6) = 0$$

$$\therefore AO = -(17{\cdot}5 + 3{\cdot}5)\frac{6}{2{\cdot}62} = -48{\cdot}08 \text{ tons.}$$

The stress in AX is found by taking moments about the point 3, the lever arm, or the perpendicular distance from 3 on AX, being 2·4 ft. :—

$$-AX(2{\cdot}4) + R(6) = 0$$

$$\therefore AX = +(17{\cdot}5 + 3{\cdot}5)\frac{6}{2{\cdot}4} = +52{\cdot}5 \text{ tons.}$$

---

[1] This example is given merely to illustrate the methods of calculation; it is unsuitable for such a small span.

The stress in the vertical member AB is found by passing a plane through AO, AB and BX, and taking moments about the intersection of AO and BX.

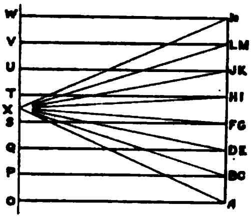


FIG. 223.

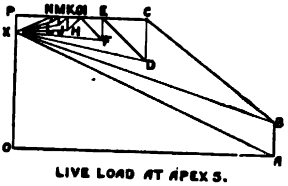


FIG. 224.

$$- AB(8{\cdot}4) - R(2{\cdot}4) = 0$$

$$\therefore AB = -(17{\cdot}5 + 3{\cdot}5)\frac{2{\cdot}4}{8{\cdot}4} = -6 \text{ tons.}$$

The stresses in the three members of the second bay may be

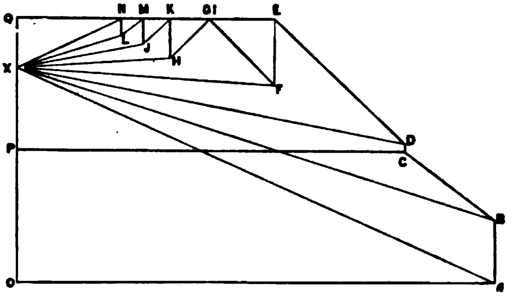


FIG. 225.

found by passing a plane through PC, BC and BX, and taking moments, thus:—

For the stress in PC take moments about the point 4, for BX take moments about the point 3, and for BC take moments about the intersection of PC and BX:—

$$4{\cdot}5PC + 12R - 6 \times 6 = 0$$

$$\therefore PC = -\frac{21 \times 12 - 36}{4{\cdot}5} = -48 \text{ tons}$$

$$-2{\cdot}5BX + 6R = 0$$

$$\therefore BX = +\frac{6 \times 21}{2{\cdot}5} = 50{\cdot}4 \text{ tons.}$$

$$5{\cdot}04BC - R \times 2{\cdot}4 + 5 \times 8{\cdot}4 + 1 \times 8{\cdot}4 = 0.$$

This equation may be written thus:—

$$
\begin{aligned}
0 &= 5{\cdot}04\mathrm{BC} - \{(\tfrac{1}{8} + \ldots + \tfrac{6}{8})2{\cdot}4 - (8{\cdot}4 - \tfrac{7}{8} \times 2{\cdot}4)\} \\
&\quad - 5\{(\tfrac{1}{8} + \ldots + \tfrac{6}{8})2{\cdot}4 - (8{\cdot}4 - \tfrac{7}{8} \times 2{\cdot}4)\} \\
\therefore 0 &= 5{\cdot}04\mathrm{BC} - (6{\cdot}3 - 6{\cdot}3) \\
&\quad - 5(6{\cdot}3 - 6{\cdot}3).
\end{aligned}
$$

This equation shows that the diagonals are unstressed when the bridge is completely loaded, either with the dead load only or with the dead and live loads. If the panel-points on the right-hand side of the section plane cutting the three bars in the bay under consideration are fully loaded, and the panel-point on the left is loaded with the dead load only, the equation becomes—

$$
\begin{aligned}
0 &= 5{\cdot}04\mathrm{BC} - (6{\cdot}3 - 6{\cdot}3) \\
&\quad - 5 \times 6{\cdot}3 \\
\therefore \mathrm{BC} &= 6{\cdot}25 \text{ tons.}
\end{aligned}
$$

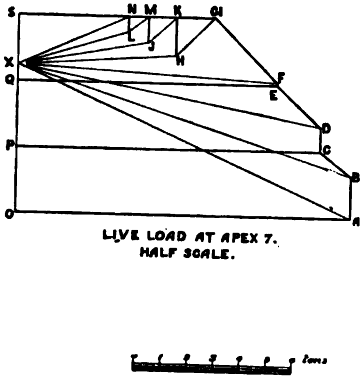


FIG. 226.

Hence the greatest tension on any diagonal occurs when the panel-points on the right are fully loaded. If the panel-point on the left is fully loaded, and those on the right with the dead load only, the equation becomes—

$$
\begin{aligned}
0 &= 5{\cdot}04\mathrm{BC} - (6{\cdot}3 - 6{\cdot}3) \\
&\quad + 5 \times 6{\cdot}3 \\
\therefore \mathrm{BC} &= -\ 6{\cdot}25 \text{ tons.}
\end{aligned}
$$

Hence the greatest compression on any diagonal occurs when the panel-points on the left are fully loaded, and those on the right with the dead load only.

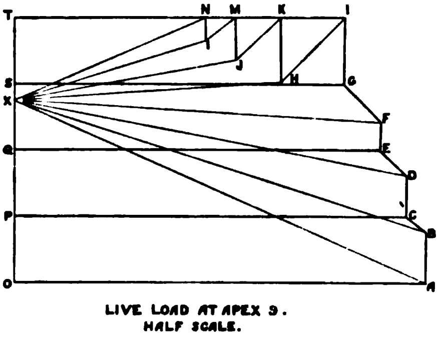


FIG. 227.

In a similar manner it can be shown that when the diagonals slope in the contrary direction, *i.e.*, from 2 to 5, instead of from 3 to 4, the same rule holds good if the word "tension" is altered to "compression," and *vice versâ*.

If both diagonals are present, they should be designed for tension only, in which case only one will be in action at a time, and the vertical members will be in compression.

In a similar manner the stresses in the remaining bars may be found and tabulated, or they may be written on the members of the girder, as in Fig. 228.

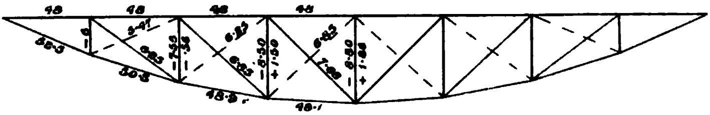


FIG. 228.

If the girder have a single system of triangulation, the stresses may be derived from Fig. 228, thus. Figs. 229 and 230.

The stresses in the bowstring truss shown in Fig. 222 may be determined by drawing reciprocal figures to scale, and measuring the lengths of the lines as explained in Chapter III. One diagram is sufficient to determine the stresses due to the dead load, but for the

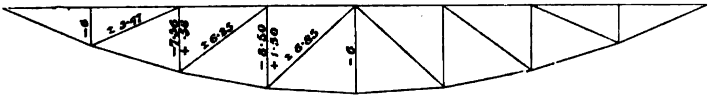


FIG. 229.

partially distributed live load it is necessary to consider the stresses for the live load at each apex separately, and then to combine the results for the total stress.

Fig. 223 shows the reciprocal figure for the bowstring truss for the dead load; Fig. 224, the live load only at the apex 3; Fig. 225, the live load at apex 5; Fig. 226, the live load at apex 7; Fig. 227, the live load at apex 9. The sum of the stresses obtained by measuring the lines in the reciprocal figures and tabulating the results

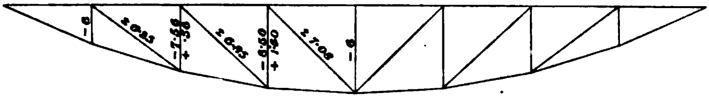


FIG. 230.

may be used as a check on the method of moments, in which case the mean value should be taken in designing the structure.

In a polygonal girder in which the panel-points in the polygonal chord do not lie on a parabolic curve, the dead load will produce stresses in the diagonals. Thus Fig. 231 shows a form of truss suitable for a timber bridge, and Fig. 232 a truss of the same dimensions suitable for an iron or steel bridge, the compression members in each case being denoted by thicker lines.

For a highway bridge with a roadway 20 ft. wide and two overhanging footways each 5 ft. wide, the dead load at each panel-point is about 14 tons, and the live load 11·2 tons, or 0·8 of the dead load.

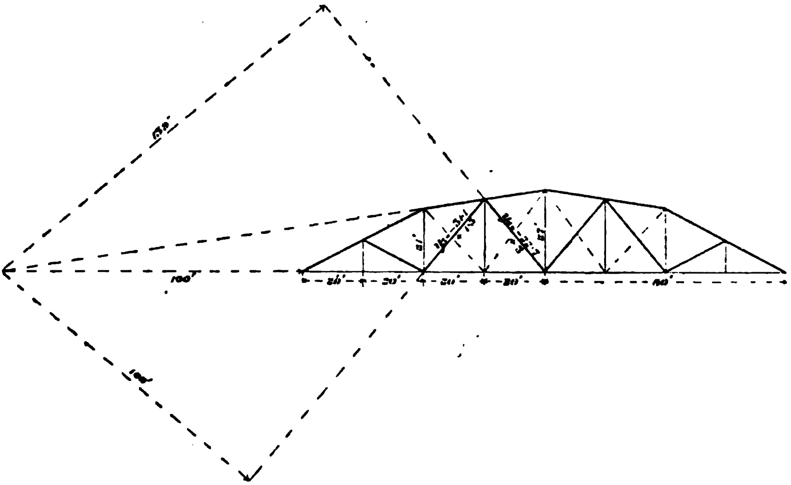


FIG. 231.

The stress in the diagonal member in the third bay, denoted by Y_3, Fig. 231, is therefore—

$$\begin{aligned} 0 = &- 108\,y_3 - [14\{(\tfrac{1}{8} + \dots + \tfrac{5}{8})100 - (120 - \tfrac{7}{8} \times 100) \\ &- (140 - \tfrac{6}{8} \times 100)\}] - 0{\cdot}8[14\{(\tfrac{1}{8} + \dots + \tfrac{5}{8})100 \\ &- (120 - \tfrac{7}{8} \times 100) - (140 - \tfrac{6}{8} \times 100)\}] \end{aligned}$$

$$\therefore\ 0 = - 108y_3 \left.\begin{aligned} - 2625 + 1365 \\ - 2100 + 1092 \end{aligned}\right\}$$

$$\begin{aligned} \therefore y_3 = &- 31{\cdot}1 \text{ tons.} \\ &- \ 1{\cdot}5 \quad ,, \end{aligned}$$

For the dead load only—

$$0 = - 108y_3 - 2625 + 1365$$

$$\therefore y_3 = - 11{\cdot}7 \text{ tons.}$$

In Fig. 184—

$$\begin{aligned} 0 = 116y_3 &- [14\{(\tfrac{1}{8} + \dots + \tfrac{5}{8})100 - (120 - \tfrac{7}{8} \times 100) \\ &- (140 - \tfrac{6}{8} \times 100)\}] - 0{\cdot}8[14\{(\tfrac{1}{8} + \dots + \tfrac{5}{8})100 \\ &- (120 - \tfrac{7}{8} \times 100) - (140 - \tfrac{6}{8} \times 100)\}] \end{aligned}$$

$$\begin{aligned} \therefore\ 0 = 116y_3 &- 2625 + 1365 \\ &- 2100 + 1092 \end{aligned}$$

$$\therefore y_3 = + 29 \text{ tons.}$$

The stresses in the diagonals of the fourth bay, Fig. 231, are—

$$
\begin{aligned}
0 = 139y_4 &- [14\{(\tfrac{1}{8} + \ldots + \tfrac{4}{8})100 - (120 - \tfrac{7}{8} \times 100) - (140 - \tfrac{6}{8} \times 100) \\
&- (160 - \tfrac{5}{8} \times 100)\}] - 0{\cdot}8[14\{(\tfrac{1}{8} + \ldots + \tfrac{4}{8})100 \\
&- (120 - \tfrac{7}{8} \times 100) - (140 - \tfrac{6}{8} \times 100) - (160 - \tfrac{5}{8} \times 100)\}] \\
0 = 139y_4 &- 1750 + 455 + 910 + 1365 \\
&- 1400 + 364 + 728 + 1092 \\
\therefore 0 = 139y_4 &- 1750 + 2730 \\
&- 1400 + 2184 \\
\therefore y_4 = &- 22{\cdot}7 \text{ tons.} \\
&+ 3{\cdot}0 \quad ,,
\end{aligned}
$$

In Fig. 232—

$$
\begin{aligned}
0 = -128y_4 &- 1750 + 2730 \\
&- 1400 + 2184
\end{aligned}
\Bigg\}
$$

$$
\begin{aligned}
\therefore y_4 &= +24{\cdot}7 \text{ tons.} \\
&\quad -3{\cdot}28 \quad ,,
\end{aligned}
$$

In a similar manner the remaining stresses may be found.

In Fig. 231 it would be desirable to introduce counterbraces in the four central panels, as indicated in the dotted lines; the largest dimensions should be at right angles to the plane of the truss for the sake of greater lateral stiffness. The diagonals should be designed for compression only, and provision should be made for taking up the shrinkage of the timber, such as by means of wedges at the joints, which would put the diagonals in initial compression. Otherwise the diagonals should be designed for both tension and compression. The tension members are conveniently constructed with a group of four iron or steel bolts, which may be screwed up as required.

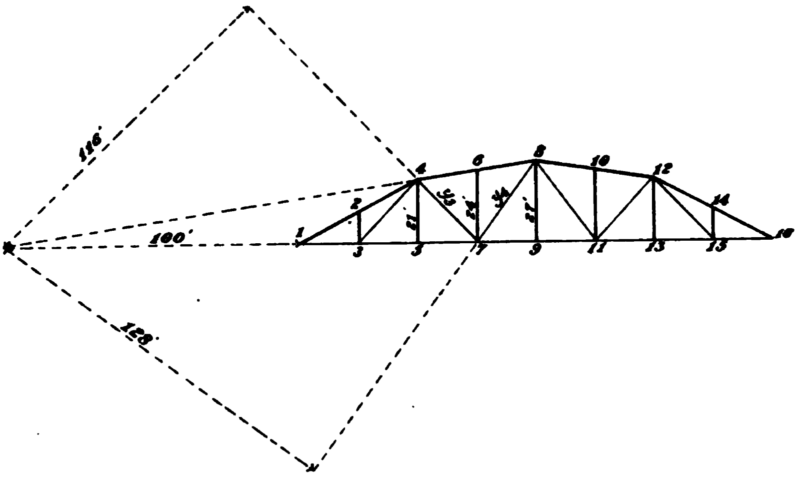


FIG. 232.

The use of wedges and counterbraces introduces some ambiguity in the determination of the stresses.

In Fig. 232 the two middle panels should be counterbraced by means of rods provided with union screws for putting the counterbrace in initial tension, or these may be omitted and the diagonals designed for compression as well as tension, which may easily be accomplished in an iron or steel bridge, thus avoiding any ambiguity in the determination of the stresses.

The stresses in these trusses might also be found by means of reciprocal figures, as illustrated with respect to the truss shown in Fig. 222.

Trusses with Subdivided Panels.—In an ordinary Pratt truss bridge of from 150 to 200 ft. span there is no necessity to provide subdivided panels, but if it is desired to use this form of truss for 400 ft. span, the panels would be inconveniently long. Fig. 233 illustrates a truss sometimes called the Baltimore truss, consisting of 16 panels, derived from a Pratt truss of 8 panels. Each panel length may conveniently be made 25 ft. by introducing sub-verticals u_1l_1, u_3l_3, u_5l_5, etc., and the sub-diagonals u_1l_2, au_4, bu_6, etc. The depth of the truss may be taken as 50 ft., so that the diagonals are inclined at 45°. The dead load may be assumed to act at the top and bottom panel-points, neglecting the portion acting at the points *a*, *b*, *c*, etc. The dead load for a bridge of 400 ft. span carrying a double line of way would be about 7200 pounds per lineal foot, and the load at the panel-points of each truss would be 90,000 pounds, or say 90 units. We may take 30 units as acting at u_1, u_2, etc., and 60 at l_1, l_2, l_3, etc., also $\sec\theta = \sqrt{2}$.

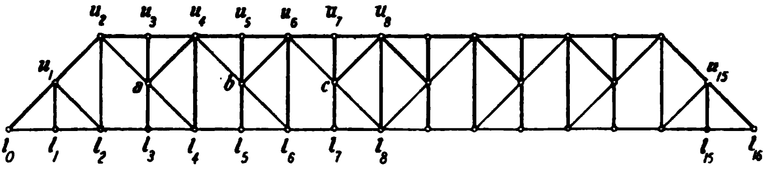


FIG. 233.

The stresses may be found in a similar manner to that used for the Pratt truss, excepting in the case of the sub-verticals and sub-diagonals. The stresses in the sub-verticals for the dead load are $u_1l_1 = 60$ units = the stress in al_3, bl_5, cl_7, etc. The stresses in u_3a, u_5b, u_7c, etc., are each equal to 30 units. The stresses in the sub-diagonals u_4a, u_6b, u_8c, etc., are each 63 units. The reactions at l_0 and l_{16} are each 675 units. The stress in u_4l_4 is found by taking a section cutting u_3u_4, u_4a, u_4l_4, and l_4l_5. Considering the portion on the left of the section, and putting the sum of the vertical components equal to zero, we have—

Stress in u_4l_4 + vertical component of the stress in u_4a + vertical shear in the section $= 0$,

or stress in $u_4l_4 = -3\frac{1}{2} \times 90 - 30 - 45 = -390$ units. The stress in al_4 is found by considering the portion of the truss to the left of a section cutting the four bars u_3u_4, u_4a, al_4 and l_3l_4. The vertical shear in the section is 405 units, the vertical component of the stress in al_4 is $405 + 45 = 450$ units, or the stress in al_4 is 636 units. In a similar manner the remaining web stresses may be found. The stresses in the chords are found by the method of chord increments, as in Chapter VIII., p. 169, thus :—

The stress in $l_2l_3 = 675$ − vertical component of the stress in $u_1l_2 = 675 - 45 = 630$ units.

The stress in $u_2u_4 = 630$ + vertical component of the stress in $u_2a = 630 + 5\frac{1}{2} \times 90 = 1125$ units.

In a similar manner the remaining chord stresses may be found.

Fig. 234 illustrates a truss derived from the Parker truss (see Plate III.) with subdivided panels, as designed by the American Bridge Company for the bridge over the Delaware River on the Pennsylvania railway. The spans of this bridge are 533 ft. in length, the sub-verticals are carried upwards to stiffen the upper chord members, and the long vertical columns are stiffened in a similar manner by means of horizontal struts *ab*, *cd*, and *ef*. This truss is a fine example of its kind, and combines the advantage of the bowstring truss in rendering the chord and web stresses more nearly uniform; at the same time it possesses all the advantages of the subdivided panels equally with the Baltimore truss, rendering it very suitable for long spans.

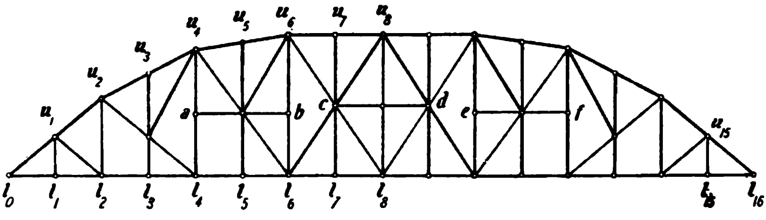


FIG. 234.

CHAPTER X.

THE SLOPE AND DEFLECTION OF BEAMS SUPPORTED AT THE ENDS—BENDING MOMENTS, SHEARING STRESSES SLOPE ; AND DEFLECTION OF BEAMS FIXED AT ONE OR BOTH ENDS.

THE investigation of the equations of slope and deflection requires a knowledge of the calculus, but the principles on which the investigation is based may be understood without such knowledge; also the results, which will be found tabulated, may be used in connection with the slope and deflection of beams as ordinary formulæ.

By referring to Fig. 122, which represents a beam deflected under the action of transverse loads, we may proceed as follows :—

Let R = the radius of curvature of the neutral axis of the beam.
a = the strain or elongation per unit of length of the fibre.
y = the distance of the extreme fibre from the neutral axis.
l = the length, measured along the neutral axis, between two planes which are parallel in the undeflected beam, but which converge towards the centre of curvature in the deflected beam.
$l + al$ = the increased length due to bending, measured along the extreme fibres.
f = the intensity of stress at the extreme fibres.
E = the modulus of elasticity.
I = the moment of inertia.
i = the slope of the beam at any point x from the origin.
v = the deflection of the beam.

Then it has been proved that—

$$\frac{\text{R} + y}{\text{R}} = \frac{l + al}{l} \quad \therefore 1 + \frac{y}{\text{R}} = 1 + a.$$

$$\therefore \frac{y}{\text{R}} = a,$$

$$\text{but } a = \frac{f}{\text{E}} \quad \therefore \frac{f}{\text{E}} = \frac{y}{\text{R}};$$

but in Chapter V. it has been proved that—

$$\text{M} = \frac{f\text{I}}{y} \quad \therefore \frac{\text{M}}{\text{I}} = \frac{f}{y}$$

$$\therefore \frac{1}{\text{R}} = \frac{f}{\text{E}y} = \frac{\text{M}}{\text{EI}}.$$

It is proved in books on the Differential Calculus that—

$$\frac{1}{R} = \frac{\frac{d^2v}{dx^2}}{\left\{1 + \left(\frac{dv}{dx}\right)^2\right\}^{\frac{3}{2}}}.$$

In the case of beams the deflection is small, so that the value of $\left(\frac{dv}{dx}\right)^2$ is so small that it may be neglected, in which case—

$$\frac{1}{R} = \frac{d^2v}{dx^2} = \frac{M}{EI}.$$

If i denote the circular measure of the slope of the beam at a distance x from the origin of co-ordinates, since—

$$i = \tan i = \frac{dv}{dx} \text{ nearly}$$

$$\frac{d^2v}{dx^2} = \frac{M}{EI},$$

$$\therefore i = \frac{dv}{dx} = \int\frac{M}{EI} \cdot dx$$

$$v = \iint\frac{M}{EI} \cdot dx_2.$$

E is a constant for the same material, and I is a constant if the section of the beam is uniform; M can also be expressed in terms of x; so that the above equations may be written—

$$i = \frac{1}{EI}\int M \cdot dx, \text{ and } v = \frac{1}{EI}\iint M \cdot dx_2.$$

The method of using these equations is illustrated in the following examples.

Case I.—Cantilever loaded at the extremity.

In this case—

$$M = -W(c - x)$$

$$\therefore i = -\frac{W}{EI}\int(c - x)dx$$

$$= -\frac{W}{EI}\left(cx - \frac{x^2}{2}\right) + C.$$

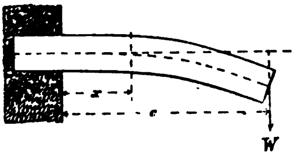


FIG. 235.

When $x = 0$, $i = 0$. $\therefore C = 0$,

$$\therefore i = -\frac{W}{EI}\left(cx - \frac{x^2}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When i is the slope at a distance x from the origin—

$$v = \int i \cdot dx = -\frac{W}{EI}\int\left(cx - \frac{x^2}{2}\right)dx = -\frac{W}{EI}\left(\frac{cx^2}{2} - \frac{x^3}{6}\right) + C.$$

When $x = 0, v = 0$. $\therefore$ C = 0,

$$\therefore v = -\frac{W}{EI}\left(\frac{cx^2}{2} - \frac{x^3}{6}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To find the greatest slope and deflection, we have only to remember that both equations (1) and (2) reach their maximum value when $x = c$; hence, denoting the greatest slope and deflection by i_0 and v_0 respectively, we have—

$$i_0 = -\frac{Wc^2}{2EI}; \text{ and } v_0 = -\frac{Wc^3}{3EI}.$$

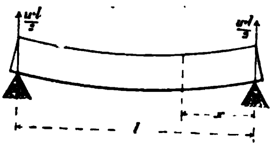


FIG. 236.

Case II.—Beam supported at both ends and loaded uniformly, the load being w per unit of length.

In this case—

$$M = \frac{wlx}{2} - \frac{wx^2}{2} = \frac{w}{2}(lx - x^2)$$

$$i = \frac{w}{2EI}\int(lx - x^2)dx = \frac{w}{2EI}\left(\frac{lx^2}{2} - \frac{x^3}{3}\right) + C.$$

Note.—If the origin of co-ordinates is taken at the centre of the beam supported at each end, the constants in the integration are equal to zero.

When $x = \frac{l}{2}$, $i = 0$.

$$\therefore 0 = \frac{w}{2EI}\left(\frac{l^3}{8} - \frac{l^3}{24}\right) + C$$

$$\therefore C = -\frac{w}{2EI}\left(\frac{l^3}{8} - \frac{l^3}{24}\right) = -\frac{wl^3}{24EI}$$

$$\therefore i = \frac{w}{2EI}\left(\frac{lx^2}{2} - \frac{x^3}{3}\right) - \frac{wl^3}{24EI} = \frac{w}{24EI}\{6lx^2 - 4x^3 - l^3\}$$

$$\therefore v = \int i \, . \, dx = \frac{w}{24EI}\int(6lx^2 - 4x^3 - l^3)dx$$

$$= \frac{w}{24EI}(2lx^3 - x^4 - l^3x) + C.$$

To find the greatest slope, put $x = 0$, or $x = l$.

$$\therefore i_0 = \frac{wl^3}{24EI} = \frac{Wl^2}{24EI}, \text{ where } W = wl.$$

For the greatest deflection, put $x = \frac{l}{2}$.

$$\therefore v_0 = \frac{w}{24EI}\left(\frac{5l^4}{16}\right) = \frac{5wl^4}{384EI} = \frac{5Wl^3}{384EI}.$$

In a similar manner, the greatest slope and deflection may be found for any beam loaded in any way whatever.

When $x = 0$; $v = 0$, $\therefore$ C = 0

and $v = \frac{w}{24EI}(2lx^3 - x^4 - l^3x)$.

In the following table $W = wl$ = the total load, will be used.

TABLE XLII.

TABLE OF MAXIMA SLOPES AND DEFLECTIONS FOR BEAMS OF UNIFORM SECTION.

Description of beam.	Greatest slope.	Greatest deflection.
Fixed at one end and loaded at the other . .	$\frac{Wl^2}{2EI}$	$\frac{Wl^3}{3EI}$
Fixed at one end and loaded uniformly . .	$\frac{Wl^2}{6EI}$	$\frac{Wl^3}{8EI}$
Supported at both ends and loaded at the centre	$\frac{Wl^2}{16EI}$	$\frac{Wl^3}{48EI}$
Supported at both ends and loaded uniformly .	$\frac{Wl^2}{24EI}$	$\frac{5Wl^3}{384EI}$

By referring to the results in the foregoing table, it will be seen that both the slope and deflection are proportional to the load, but that the slope is proportional to the square of the length, while the deflection is proportional to the cube of the length. Hence we may express the slope and deflection thus:—

$$i_0 = \frac{nWl^2}{EI} \quad v_0 = \frac{mWl^3}{EI}$$

where n and m have the numerical values given in the foregoing table; thus in the case of a beam supported at both ends and loaded uniformly, $n = \frac{1}{24}$, and $m = \frac{5}{384}$.

It is frequently convenient to express the slope and deflection of a beam in terms of the maximum intensity of working stress; thus in the case of a beam supported at both ends and loaded uniformly—

$$\frac{Wl}{8} = \frac{fI}{y} \qquad \therefore W = \frac{8fI}{ly}.$$

$$\text{Hence the slope } i_0 = \frac{fl}{3Ey}$$

$$\text{and the deflection } v_0 = \frac{5fl^2}{48Ey}.$$

In this case the slope is proportional to the length, while the deflection is proportional to the square of the length.

In a similar manner, the slopes and deflections of the other three cases may be expressed in terms of the maximum intensity of working stress. Hence we may express i_0 and v_0 thus:—

$$i_0 = \frac{n'fl}{Ey}, \text{ and } v_0 = \frac{m'fl^2}{Ey}$$

In rectangular beams—

$$I = \frac{1}{12}bd^3 \qquad y = \frac{d}{2}.$$

By substituting these values in the foregoing table, we obtain formulæ applicable to rectangular beams; thus in the case of a beam supported at both ends and loaded uniformly—

$$i_0 = \frac{Wl^2}{2Ebd^3}, \text{ and } v_0 = \frac{5Wl^3}{32Ebd^3}.$$

If we denote the intensity of stress at the extreme fibre by f, we have—

$$i_0 = \frac{2fl}{3Ed}, \text{ and } v_0 = \frac{5fl^2}{24Ed}.$$

In a similar manner we may find the value of i_0 and v_0 for the other three cases.

Beams of Uniform Strength.—In a beam of uniform strength the moment of resistance of any cross-section is proportional to the bending moment at that cross-section.

In Figs. 237 to 244 we have various forms in which either the breadth or the depth is constant, except in Fig. 239, which is a beam of similar cross-section. Figs. 237, 238 and 239 represent cantilevers of uniform strength when loaded at the extremity with a concentrated load; and Figs. 240, 241 and 242 represent cantilevers loaded with a uniform load.

FIG. 237.

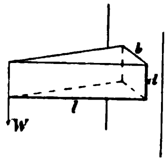


FIG. 238.

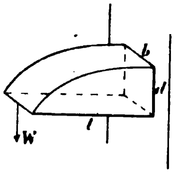
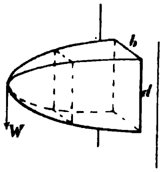


FIG. 239.



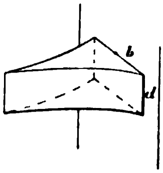


FIG. 240.

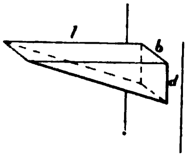


FIG. 241.

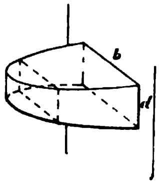


FIG. 242.

Fig. 243 shows the shape of a beam of uniform strength, supported at both ends and loaded in the centre when the depth is constant.

Fig. 244 shows a beam when the breadth is constant.

Fig. 245 shows the shape of a beam for similar cross-section.

In practice the above forms could not be carried out, as it is necessary to provide for the resistance to the shearing stresses as well as the bending moments; consequently they may be modified as shown in Figs. 246 and 247, where the sectional areas over the supports are in accordance with the shearing stresses.

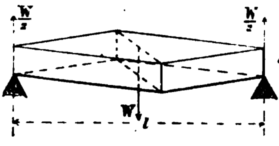


FIG. 243.

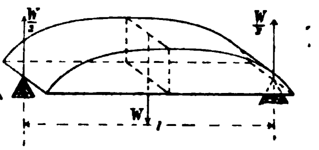


FIG. 244.

In beams of uniform strength the moment of inertia is not constant, and the term $\frac{M}{EI}$ contains two variables.

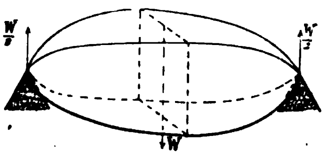


FIG. 245.

In the case of Fig. 237, d is constant, and b varies; here, if b_1 denote the breadth of any section, $b_1 = b\frac{x}{l}$.

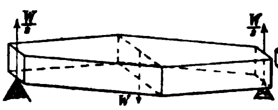


FIG. 246.

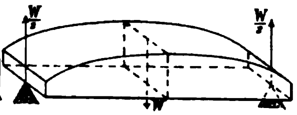


FIG. 247.

If $b = b_1$, and d varies, as in Fig. 238, then $d_1^2 = d^2\frac{x}{l}$, where d_1 denotes the depth at any section. If both b and d vary, as in Fig. 239—

$$\frac{b}{d} = \frac{b_1}{d_1} \qquad \therefore b_1 = \frac{bd_1}{d}.$$

In Fig. 237—

$$Wx = \frac{2fI}{d_1} \qquad \therefore f = \frac{Wxd_1}{2I}$$

where $d_1 = 2y$; but in rectangular sections—

$$f = \frac{6Wx}{b_1d_1^2} = \frac{6Wl}{bd^2}$$

since f is a constant;

$$\therefore \frac{b_1d_1^2}{bd^2} = \frac{x}{l}.$$

$$\text{Also } \frac{d^2v}{dx_2} = -\frac{M}{EI} = -\frac{Wx}{\frac{1}{12}Eb_1d_1^3}$$

where b and d vary.

If the height is constant and equal to d, then $b_1 = \frac{bx}{l}$,

$$\therefore \frac{d^2v}{dx^2} = -\frac{12Wl}{Ed^3b}.$$

Integrating this expression, and remembering that $\frac{dv}{dx} = 0$ when $x = l$, we have—

$$\frac{dv}{dx} = -\frac{12Wlx}{Ed^3b} + \frac{12Wl^2}{Ed^3b}.$$

Integrating again, and remembering that $v = 0$ when $x = l$, we have—

$$v = -\frac{6Wlx^2}{Ed^3b} + \frac{12Wl^2x}{Ed^3b} - \frac{6Wl^3}{Ed^3b}.$$

The maximum deflection is $v_0 = \frac{6Wl^3}{Ed^3b}$, whereas for a uniform cross-section we found $v_0 = \frac{4Wl^3}{Ed^3b}$, or the beam of uniform strength deflects one and a half times as much as the beam of uniform cross-section.

In a similar manner, it may be proved that the beam of constant breadth (Fig. 238) deflects $v_0 = \frac{8Wl^3}{Ed^3b}$, or twice as much as the beam of uniform cross-section.

Also, for similar cross-section (Fig. 239), we have $v_0 = \frac{36Wl^3}{5Ed^3b}$.

For any cross-section, using the general formulæ—

$$i_0 = \frac{nWl^2}{EI}, \text{ and } v_0 = \frac{mWl^3}{EI},$$

we have the following values for n and m:—

TABLE XLIII.

TABLE OF VALUES FOR n AND m FOR BEAMS OF UNIFORM STRENGTH.

Description of beam.—Uniform depth.	n.	m.
Fixed at one end and loaded at the other	1	$\frac{1}{2}$
Fixed at one end and loaded uniformly	$\frac{1}{2}$	$\frac{1}{4}$
Supported at both ends and loaded at the centre . .	$\frac{1}{8}$	$\frac{1}{32}$
Supported at both ends and loaded uniformly . . .	$\frac{1}{16}$	$\frac{1}{64}$

TABLE XLIV.

Description of beam.—Uniform breadth.	*n.*	*m.*
Fixed at one end and loaded at the other . . .	2	$\frac{2}{3}$
Supported at both ends and loaded at the centre . .	$\frac{1}{4}$	$\frac{1}{24}$
Supported at both ends and loaded uniformly . .	0·098	0·018

Deflection of Beams under Loads Developing a Known Intensity of Stress.—In this case we use the formula—

$$v_0 = \frac{mfl^2}{Ey}.$$

f denotes the maximum intensity of stress on the weaker side of the beam.

If f_t denote the maximum intensity of tensile stress, and f_c the maximum intensity of compressive stress, then $f = \frac{f_t + f_c}{2}$. If d denote the depth of the beam—

$$v_0 = \frac{m(f_t + f_c)l^2}{Ed}.$$

This formula may be applied to wrought-iron or steel girders. The values of m are given in the following table :—

TABLE XLV.

No.	Uniform cross-section.	*m.*
1	Cantilever fixed at one end and loaded at the other . .	$\frac{1}{3}$
2	Cantilever fixed at one end and loaded uniformly . . .	$\frac{1}{4}$
3	Beam supported at both ends and loaded in the centre . .	$\frac{1}{12}$
4	Beam supported at both ends and loaded uniformly . .	$\frac{5}{48}$
	Uniform Strength and Depth.	
5	Under either of the conditions Nos. 1 to 4	$\frac{1}{8}$
	Uniform Strength and Breadth.	
6	Cantilever fixed at one end and loaded at the other . .	$\frac{2}{3}$
7	Cantilever fixed at one end and loaded uniformly . . .	1
8	Beam supported at both ends and loaded at the centre . .	$\frac{1}{6}$
9	Beam supported at both ends and loaded uniformly . .	·1426

The Law of Similarity and Transverse Tests.—The law of similarity applied to transverse tests may be expressed as follows according to Martens:[1] Beams of the same material, and of similar cross-section and shape, when supported or fixed, and loaded in the same manner, require loads proportional to their sectional area to produce similar deflections.

---

[1] *Handbook of Testing Materials*, Prof. Martens, Wiley & Sons.

The dimensions of the bearings must be similar, if the conditions of the law are to be strictly complied with.

To establish this law, let us assume two bodies, A and A_1, geometrically similar and of prismatic section. In these two bodies the following ratios exist:—

(1) lengths $\frac{d}{d_1}=\frac{b}{b_1}=\frac{l}{l_1}=\frac{1}{n}$.

(2) surfaces $\frac{db}{d_1b_1}=\frac{1}{n^2}$.

(3) volumes and weight = — $\frac{V}{V_1}=\frac{W}{W_1}=\frac{1}{n^3}$.

(4) moments of inertia $\frac{I}{I_1}=\frac{1}{n^4}$.

When deflections are indicated by ratio of deflection, a similar deformation, A and A_1, is produced when—

$$\frac{v}{l}=\frac{v_1}{l_1}\,;\ \frac{v}{v_1}=\frac{l}{l_1}=\frac{1}{n}.$$

For body A, representing a beam supported at each end and loaded in the centre with a load W—

$$v=\frac{Wl^3}{48EI}\quad\therefore\frac{v}{l}=\frac{Wl^2}{48EI}.$$

For body A—

$$\frac{v_1}{l_1}=\frac{W_1l_1^2}{48EI_1}=\frac{W_1l^2n^2}{48EIn^4}=\frac{W_1l^2}{48EIn^2}$$

Therefore
$$\frac{vl_1}{v_1l}=1=\frac{Wl^2}{48EI}\times\frac{48EIn^2}{W_1l^2}=\frac{W}{W_1}n^2.$$

Hence the same ratio of deflection, *i.e.*, geometrically similar deformations, will be produced by loads which are as 1 to n^2, or as the cross-sectional areas.

The relation of stress may be found thus:—

$$f=\frac{Wl}{4}\frac{y}{I}$$

Since $W_1=n^2W$, $l_1=nl$, $y_1=ny$, $I_1=n^4I$,

$$\therefore f_1=\frac{W_1l_1}{4}\frac{y_1}{I_1}=\frac{Wlyn^4}{4In^4}=\frac{Wly}{4I}$$

$$\therefore\frac{f}{f_1}=1.$$

Hence: Equal stresses produce equal ratios of deflection in bodies geometrically similar. So that if stress f_1 and ratios of deflection v/l

be plotted for geometrically similar bodies, the curves obtained will coincide.

Deflection due to Shearing.—Let v_s be the deflection due to shearing, and let ϕ denote the transverse shearing strain for a unit length of beam. The transverse strain for an indefinitely short portion dx of the neutral surface will be—

$$dv_s = \phi dx$$

$$\text{since } \phi = \frac{\sigma_s}{C}$$

$$dv_s = \phi dx = \frac{\sigma_s}{C}dx$$

$$\text{also since } \sigma_s(\text{max.}) = \frac{Sy^2}{2I}$$

$$dv_s = \frac{Sy^2}{2IC}dx$$

where y is the distance of the extreme fibre from the neutral axis.

The general expression for the deflection due to shearing is—

$$v_s = \frac{y^2}{2IC}\int Sdx.$$

In the case of a beam supported at each end and loaded uniformly with w per unit of length. Here $S = wx$—

$$v_s = \frac{wy^2}{2IC}\int_0^{\frac{l}{2}} xdx = \frac{wy^2l^2}{16IC}.$$

If the beam is rectangular in cross-section—

$$y = \frac{d}{2}, I = \frac{bd^3}{12} \text{ and } v_s = \frac{3wl^2}{16bdC}.$$

The deflection due to direct stresses has been shown to be—

$$v_0 = \frac{5wl^4}{384EI}.$$

Therefore the total deflection in the centre due to direct stresses and shear is—

$$v_s + v_0 = \frac{wy^2l^2}{16IC} + \frac{5wl^4}{384EI}.$$

Elastic Resilience due to Bending in Transverse Tests.—The resilience due to the direct tensile and compressive stresses due to bending may be found thus:—

At any distance y from the neutral layer of a beam, consider a differential length, dl, subjected to an intensity of stress σ, then the elastic change is—

$$\frac{\sigma}{E}dl.$$

If dA is an indefinitely small portion of the normal sectional area, A, the average stress acting through the small change of length is—

$$\frac{\sigma dA}{2} = \frac{My}{2I}dA.$$

Hence the work performed in any normal section for which M remains unchanged is—

Since $\int \sigma dAy = M$

$$\int \frac{M}{2IE}\sigma y \,.\, dA \,.\, dl = \frac{M^2}{2EI}dl.$$

The work performed throughout the whole beam is therefore—

$$\int \frac{M^2}{2EI}dl.$$

If in a uniformly loaded beam supported at each end—

$$M = \frac{w}{2}(lx - x^2)$$

and since $dl = dx$

$$\text{Resilience} = \frac{1}{2EI}\int_0^l \frac{w^2}{4}(l^2x^2 - 2lx^3 + x^4)dx$$

$$= \frac{w^2l^5}{240EI} = \frac{W^2l^3}{240EI}.$$

The equation gives the value of the total work performed by the direct tensile and compressive stresses in a beam supported at each end, and loaded uniformly, assuming that the moment of inertia of the cross-section is constant. If the beam is loaded at the centre, the value of the bending moment at any point is—

$$M = \frac{W}{2}x$$

$$\text{Resilience} = \frac{1}{2EI} \,.\, 2\int_0^{\frac{l}{2}} \frac{W^2}{4}x^2dx = \frac{W^2l^3}{96EI}.$$

Resilience due to Shearing Stresses.—The work performed by the vertical shearing stresses in a deflected beam may be found in the following manner:—

Let S denote the total transverse shearing stress, I, the moment of inertia of the cross-section about the neutral axis, b, the width of the beam, σ_s, the intensity of shearing stress at any point in the cross-section at a distance y from the neutral axis, then—

$$\sigma_s = \frac{S}{2I}\left(\frac{d^2}{4} - y^2\right).$$

$$\phi = \frac{\sigma_s}{C} = \frac{S}{2CI}\left(\frac{d^2}{4} - y^2\right).$$

The amount of shearing stress on the indefinitely small portion of the section bdy will be—

$$\sigma_s bdy,$$

and its strain in performing the work will be—

$$\phi dx,$$

x being the horizontal distance of the section from the origin, *i.e.*, from the end of the span of the beam in this case.

The differential work performed in the section will be—

$$\sigma_s bdy(\phi dx) = \frac{S^2b}{4I^2C}\left(\frac{d^2}{4} - y^2\right)^2 dxdy.$$

If the section is rectangular of breadth b, and depth d, and x and y are independent variables, integrate first with reference to y—

$$\frac{b}{4I^2C}\int S^2dx\int_{-\frac{d_1}{2}}^{\frac{d}{2}}\left(\frac{d^2}{4} - y^2\right)^2 dy = \frac{b}{4I^2C}\int S^2dx\int_{-\frac{d_1}{2}}^{\frac{d}{2}}\left(\frac{d^4}{16} - \frac{d^2y^2}{2} + y^4\right)dy$$

$$= \frac{b}{4I^2C}\int S^2dx\left(\frac{d^5 + d_1^5}{60}\right) = \frac{b(d^5 + d_1^5)}{240I^2C}\int S^2dx.$$

Since in a rectangular cross-section—

$$I = \frac{bd^3}{12} \text{ and } d = d_1,$$

$$\text{Resilience (due to shear)} = \frac{6}{5bdC}\int S^2dx.$$

The total resilience due to bending and shear is—

$$\int\frac{M^2}{2EI}dx + \frac{6}{5bdC}\int S^2dx.$$

If $wl = W$ in a rectangular beam supported at each end and uniformly loaded—

$$\int\frac{M^2}{2EI}dx = \frac{W^2l^3}{240EI} \text{ and } \frac{6}{5bdC}\int S^2dx = \frac{6w^2}{20bdC}\cdot 2\int_0^{\frac{l}{2}}(l^2 - 4lx + 4x^2)dx.$$

Since $S = \frac{w}{2}(l - 2x)$

$$\therefore \frac{6}{5bdC}\int S^2dx = \frac{w^2l^3}{10bdC} = \frac{W^2l}{10bdC}.$$

The total resilience due to direct and shearing stresses for a beam of rectangular section loaded uniformly is—

$$W^2\left(\frac{l^3}{240EI} + \frac{l}{10bdC}\right).$$

Combined Bending and Direct Stress.—Let W denote the uniformly distributed load due to the weight of the member ab (Fig.

248), and v_o the central deflection. It has been shown that the resilience of the beam due to bending is—

$$\frac{W^2l^3}{240EI}.$$

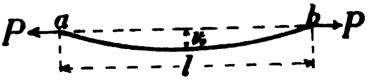


FIG. 248.

If the centre of gravity of the load W descends through a distance v_1, when the centre deflection is v_o the resilience is Wv_1, and—

$$Wv_1 = \frac{W^2l^3}{240EI}$$

$$v_1 = \frac{Wl^3}{240EI}.$$

$$\text{But since } v_o = \frac{5Wl^3}{384EI} \text{ see p. 194.}$$

$$\frac{v_1}{v_o} = \frac{384}{5 \times 240} = \frac{8}{25}; \therefore v_1 = \frac{8v_o}{25},$$

$$\text{the resilience} = \frac{8Wv_o}{25}.$$

$$\text{But since } W = \frac{384EI}{5l^3}v_o,$$

the resilience may be expressed—

$$\frac{3072EI}{125l^3}v_o^2.$$

If the actual centre deflection is v_1, the resilience of the beam will be—

$$\left(\frac{v_1}{v_o}\right)^2 \frac{W^2l^3}{240EI}.$$

If l = the length of the span, and v_o the deflection, the length of the curve is—

$$l\left(1 + \frac{8}{3}\frac{v_o^2}{l^2} - \frac{32v_o^4}{l^4} + \ldots\right),$$

but since the curvature is small we may neglect all the terms after $\frac{v_o^2}{l^2}$, and write for the length of the curve of span l—

$$l\left(1 + \frac{8}{3}\frac{v_o^2}{l^2}\right).$$

In a similar manner if l_1 is the span and v_1 the corresponding deflection, the length of the curve is—

$$l_1\left(1 + \frac{8}{3}\frac{v_1^2}{l_1^2}\right).$$

Since these lengths are sensibly equal—

$$l - l_1 = \tfrac{8}{3}\left(\frac{v_o^2}{l} - \frac{v_1^2}{l_1}\right).$$

The value of $l - l_1$ = the lengthening of the member if it is in tension, and the shortening if it is in compression under the direct stress P. In the tension member the pull P reduces a part of the deflection v_o, and in doing so lifts the weight W of the member; the remaining work of lifting W is performed by the elastic effort of the member to straighten itself from v_o to v_1, which is expressed by the quantity—

$$\frac{W^2l^3}{240EI}\left(1 - \frac{v_1^2}{v_o^2}\right).$$

Hence the equation of work may be stated in the following manner:—

$$\frac{8P}{6}\left(\frac{v_o^2 - v_1^2}{l}\right) + \frac{W^2l^3}{240EI}\left(1 - \frac{v_1^2}{v_o^2}\right) = \frac{8}{25}W(v_o - v_1)$$

$$\therefore v_1 = \frac{v_o}{1 + \frac{6}{25}\frac{Wl}{P}\cdot\frac{1}{v_o}}$$

$$\text{where } v_o = \frac{5Wl^3}{384EI}.$$

The resultant bending moment at the centre of the member is—

$$M = \frac{Wl}{8} - Pv_1.$$

If A denote the area of the member, σ the maximum intensity of stress in it, then—

$$\sigma = \frac{P}{A} + \frac{My}{I}.$$

For rectangular sections—

$$A = bd,\ y = \frac{d}{2};\ I = \frac{1}{12}bd^3;$$

$$\sigma = \frac{P}{bd} + \frac{6M}{bd^2} = \frac{1}{bd}\left(P + \frac{6M}{d}\right).$$

In a compression member it can be shown that—

$$M = \frac{Wl}{8} + Pv_1.$$

It is generally sufficiently accurate in the treatment of combined bending and direct stress in eye bars and chords of bridges to use the approximate method illustrated with reference to the Pratt truss, Plate II*a* and II*b*.

The Principle of Work Applied to the Deflection of Beams or Trusses.—The principle of work may be applied to the deflection of beams and trusses, also to the determination of statically indeterminate structures; it is more conveniently applied to the calculations of the deflection of plate-web girders than the equation of the elastic curve, so that a brief account of the method will not be out of place here. In the first instance, the method will be developed with reference to a truss, and afterwards modified to suit the case of a girder with a solid plate web.

Deflection of a Truss.—If a load gradually increasing from zero to W is applied to a truss causing a deflection v, the external work performed by the load is—

$$\frac{Wv}{2}.$$

If several loads are applied to a truss W_1, W_2, W_3, etc., causing deflections v_1, v_2, v_3, etc., respectively, the external work caused by these loads is—

$$\frac{1}{2}\left(W_1v_1 + W_2v_2 + W_3v_3 + \text{etc.}\right).$$

The internal work developed in the truss in resisting the stresses in the various members is found by multiplying the stress by the elongation in each member. Let S denote the total stress and λ the total elongation, l the length, and A the area of a member, then the internal work, or energy stored, may be expressed thus:—

$$\frac{1}{2}\left(S_1\lambda_1 + S_2\lambda_2 + S_3\lambda_3 + \text{etc.}\right),$$

and since $\lambda = \frac{Sl}{AE}$ then the internal work is—

$$\frac{1}{2}\Sigma\frac{S^2l}{AE}.$$

In a truss where the lengths, areas and stresses for each member are known, the elastic deformations can be readily calculated.

Since the internal and external work are equal, a load W upon a truss producing a deflection v beneath it may be expressed in regard to the principle of work thus:—

$$\frac{Wv}{2} = \frac{1}{2E}\Sigma\frac{S^2l}{A}$$

$$\therefore v = \frac{1}{WE}\Sigma\frac{S^2l}{A}.$$

In this manner the deflection of a truss due to a single load placed where the deflection is required may be determined. The deflection of a truss due to several loads may be found thus: In the first place the stresses in the various members must be calculated for the particular loading, then a definite portion of the total load is supposed to be placed at the apex where the deflection is desired. Let this load be denoted by w, which is usually taken as 1 pound, or 1 ton, according to the units adopted. The external work due to w is $\frac{wv}{2}$, and the internal work developed by the various loads applied on any member whose stress is S_1 is $\frac{1}{2}S_1(\epsilon_1 + \epsilon_1' \epsilon_1'' + \text{etc.})$, where ϵ_1 is the change of length due to w, and ϵ_1', ϵ_1'', etc., are the changes of length due to the other loads. The internal work in the given member due to the load w is $\frac{1}{2}S_1\epsilon_1$. Let σ_1 denote the stress in the given member due to w, then we have—

$$\epsilon_1 = \frac{\sigma_1 l_1}{A_1 E}.$$

The internal work in this member is therefore—

$$\frac{S_1\sigma_1 l_1}{2A_1E},$$

and for the whole truss—

$$\Sigma\frac{S\sigma l}{2AE}.$$

The deflection at any point in a truss for the load w may now be written—

$$v = \frac{1}{wE}\Sigma\frac{S\sigma l}{A}.$$

Deflection of a Beam.—Let f denote the extreme fibre stress distant y from the neutral axis, also f_1 the stress at a distance y_1, then $f_1 = \frac{fy_1}{y}$. The stress at a distance y_1 from the neutral axis due to a load $w = 1$ is—

$$f_1' = \frac{f'y_1}{y}.$$

Since these stresses act over an indefinitely small area denoted by A, the stress S in the above expression for v must be replaced by $\frac{Afy_1}{y}$, and the stress σ by $\frac{Af'y_1}{y}$, so that we have—

$$v = \frac{1}{E}\Sigma\frac{Afy^2_1f'l}{y^2}.$$

Let M denote the bending moment due to the loading, and m the bending moment due to $w = 1$, then $f = \frac{My}{I}$, and $f' = \frac{my}{I}$, and since $I = \Sigma Ay^2_1$ the above expression becomes, if we write dx for l—

$$v = \int\frac{Mm}{EI}dx.$$

If l denote the span of the beam—

$$v = \int_0^l \frac{Mm}{EI}dx.$$

This is the general formula for the deflection of any point of a beam of constant cross-section, and was first derived by Prof. Fraenkel. It is a special case of Castigliano's theorem.

The above expression is derived by C. W. Hudson[1] in an able paper read before the American Society of Civil Engineers and applied to find the horizontal deflection of the upper corner of a simple rolled steel beam, loaded at the centre with a load W. (Fig. 249.)

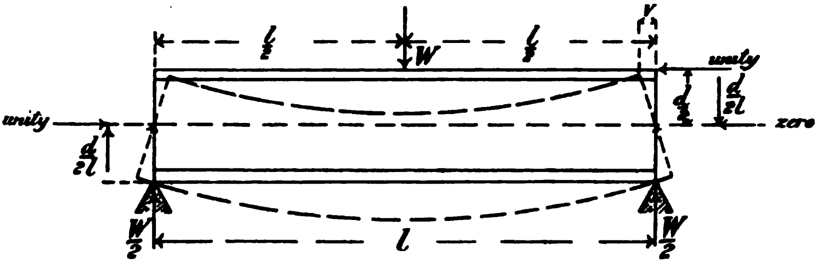


Fig. 249.

Taking each half of the beam separately, and taking the ends of the neutral axis as origin of x, we have—

$$v = \int_0^l \frac{Mmdx}{EI} = \overset{\text{Left half.}}{\int_0^{\frac{l}{2}} \frac{Mmdx}{EI}} + \overset{\text{Right half.}}{\int_0^{\frac{l}{2}} \frac{Mmdx}{EI}}$$

in which $M = \frac{Wx}{2}$ for both right and left halves of the beam, $m = \frac{dx}{2l}$ for the left, and $-\frac{d}{2l}x + \frac{d}{2}$ for the right half. Substituting these values for M and m in the general equation, we have—

$$v = \int_0^{\frac{l}{2}} \frac{Wdx^2}{4lEI}dx - \int_0^{\frac{l}{2}} \frac{Wdx^2}{4lEI}dx + \int_0^{\frac{l}{2}} \frac{Wdx}{4EI}dx$$

$$= \int_0^{\frac{l}{2}} \frac{Wdx}{4EI}dx = \frac{Wdl^2}{32EI}.$$

Mr. Hudson applies this expression to a rolled beam 20 in. deep weighing 192 pounds per yard, where W = 29,000 pounds, $l = 360$ in. and I = 1140 (inches $^{4\text{th}}$), then—

$$v = \frac{29{,}000 \times 20 \times (360)^2}{32 \times 29{,}000{,}000 \times 1140} = \frac{1}{14} \text{ in. nearly.}$$

The following example selected from several is worked out by Mr. Hudson and illustrates the method of finding the deflection of a plate web girder of variable moment of inertia :—

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[1] "Deflection of Beams with Variable Moments of Inertia," *Trans. Am. Soc. C. E.*, May, 1903.

Find the vertical deflection at the centre of a plate web girder supported at each end and loaded uniformly; the girder has three cover plates as shown in Fig. 250, consequently there are four values for the moment of inertia.

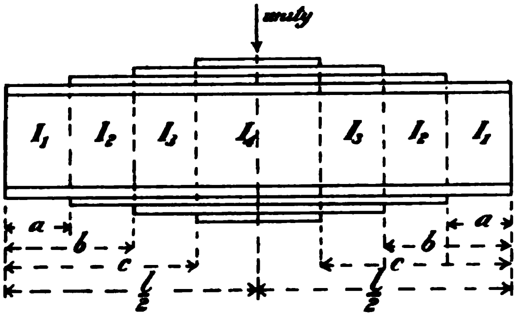

FIG. 250.

Let w denote the load per unit of length. The formula for v will have four parts—

$$v = \int_0^l \frac{\mathrm{M}m\,dx}{\mathrm{EI}} = 2\int_0^{\frac{l}{2}} \frac{\mathrm{M}m\,dx}{\mathrm{EI}} = \frac{w}{2\mathrm{E}}\int_0^a \frac{(lx^2 - x^3)dx}{\mathrm{I}_1} + \frac{w}{2\mathrm{E}}\int_a^b \frac{(lx^2 - x^3)dx}{\mathrm{I}_2}$$

$$+ \frac{w}{2\mathrm{E}}\int_b^c \frac{(lx^2 - x^3)dx}{\mathrm{I}_3} + \frac{w}{2\mathrm{E}}\int_c^{\frac{l}{2}} \frac{(lx^2 - x^3)dx}{\mathrm{I}_4}.$$

$$= \frac{w}{2\mathrm{E}}\left\{\frac{la^3}{3\mathrm{I}_1} - \frac{a_4}{4\mathrm{I}_1} + l\frac{(b^3 - a^3)}{3\mathrm{I}_2} - \frac{b^4 - a^4}{4\mathrm{I}_2} + l\frac{(c^3 - b^3)}{3\mathrm{I}_3} - \frac{c_4 - b_4}{4\mathrm{I}_3}\right.$$

$$\left. + l\frac{(l^3 - 8c^3)}{24\mathrm{I}_4} - \frac{l^4 - 16c^4}{64\mathrm{I}_4}\right\}$$

$$= \frac{5}{384} \times \frac{wl^4}{\mathrm{EI}_4} + \frac{wl}{6\mathrm{E}}\left(\frac{a^3}{\mathrm{I}_1} - \frac{a^3}{\mathrm{I}_2} + \frac{b^3}{\mathrm{I}_2} - \frac{b^3}{\mathrm{I}_3} + \frac{c^3}{\mathrm{I}_3} - \frac{c^3}{\mathrm{I}_4}\right)$$

$$- \frac{w}{8\mathrm{E}}\left(\frac{a^4}{\mathrm{I}_1} - \frac{a^4}{\mathrm{I}_2} + \frac{b^4}{\mathrm{I}_2} - \frac{b^4}{\mathrm{I}_3} + \frac{c^4}{\mathrm{I}_3} - \frac{c^4}{\mathrm{I}_4}\right)$$

which is the expression for the desired deflection. If $\mathrm{I}_1 = \mathrm{I}_2 = \mathrm{I}_3 = \mathrm{I}_4$, that is if the moment of inertia is constant, the expression becomes—

$$v = \frac{5wl^4}{384\mathrm{EI}},$$

which has already been derived from the elastic curve, page 194.

Shearing Stresses.—In order to determine the effect of the shearing stresses in producing deflection of beams, Mr. Hudson proceeds as follows:—

Consider a portion of a beam in equilibrium under the action of transverse loading as indicated in Fig. 251. If W be the resultant in position, direction and amount of all the forces to the right of the section mn, then it is clear that the portion of the beam under consideration would be kept in equilibrium by the couple Fa, and the shear S. In general, no matter what may be the actual distribution of the stresses on any section of a beam under a given loading, these stresses may be resolved into an equivalent couple and shear. The effect of the couple in doing work has been determined in accordance with the common theory of flexure. Going back to the general formula for the expression of the internal work, we have—

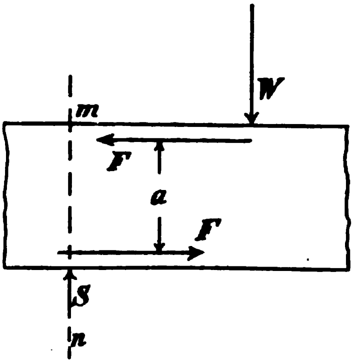

FIG. 251.

$$v = \int_0^l \frac{\mathrm{M}m\,dx}{2\mathrm{EI}} \text{ (for bending) } + \int_0^l \frac{\mathrm{S}s\,dx}{2\mathrm{CA}} \text{ (for shearing).}$$

To derive the expression for the work done by the shearing stresses—

Let S = shear at any point, due to a given loading.

Let s = shear at any point, due to a load of unity at any point where the deflection is desired.

A = total area of the cross-section.

C = the coefficient of elasticity for shear or coefficient of rigidity, and let dx be an infinitesimal portion of the length of the beam; that is, let it be considered as the distance between two consecutive sections of the beam.

Then under the action of the shearing stress, S, the two consecutive sections will have a relative motion of $\frac{\mathrm{S}dx}{\mathrm{AC}}$, if we assume that the shearing stress is distributed uniformly over the cross-section. Now as for the bending stresses we will suppose the force of unity to be applied gradually; then the work done by the shearing stress on an elementary portion of the beam, due to the force of unity is $\frac{\mathrm{S}dx}{\mathrm{AC}} \times \frac{s}{2}$, and over the entire beam the work is—

$$\int_0^l \frac{\mathrm{S}s\,dx}{2\mathrm{CA}}.$$

Making the expression for the external and internal work equal, we have—

$$\frac{v}{2} = \int_0^l \frac{\mathrm{M}m\,dx}{2\mathrm{EI}} + \int_0^l \frac{\mathrm{S}s\,dx}{2\mathrm{CA}}, \text{ and}$$

$$v = \int_0^l \frac{\mathrm{M}m\,dx}{\mathrm{EI}} + \int_0^l \frac{\mathrm{S}s\,dx}{\mathrm{CA}}.$$

The shearing stresses are not uniformly distributed over the section, and hence the term for the work of the shearing stresses should be modified by a coefficient. The value of this coefficient for a rectangular cross-section may be shown to be $\frac{6}{5}$. For other sections the expression is not so simple, but Mr. Hudson has shown that in a plate-web girder the amount of the work done on the flanges is a very small part of the total work on the cross-section, and it is sufficiently exact to consider that the web alone resists all the shearing stresses. The effect of this assumption is to give deflections a little too great. In girders the vertical flange plates, and also the vertical legs of angles must take considerable shear, but if we neglect this also we may write the general equation for the deflection of a plate-girder of varying cross-sections of this description thus:—

$$v = \int_0^l \frac{Mm\,dx}{EI} + \int_0^l \frac{Ss\,dx}{CA_w},$$

in which A_w denotes the area of the web. Putting $C = \frac{2}{5}E$, we have—

$$v = \int_0^l \frac{Mm\,dx}{EI} + \frac{5}{2}\int_0^l \frac{Ss\,dx}{EA_w},$$

and for an ordinary plate-web girder consisting of plates and angles, with no vertical flange plates, we multiply the expression for shear by the coefficient $\frac{6}{5}$, which alters the value of v, thus:—

$$v = \int_0^l \frac{Mm\,dx}{EI} + 3\int_0^l \frac{Ss\,dx}{EA_w}.$$

In the former case the neglect of the shearing resistance of the vertical flange plates is assumed to balance the omission of the coefficient $\frac{6}{5}$. As pointed out by Prof. Merriman[1]: "The common theory of beams in which the stresses are represented by horizontal tensions and compressions and by vertical shears is an approximation only. At any point in the beam there really act tensile and compressive stresses at right angles to each other, but these are not horizontal, except at the upper and lower surfaces, while the shearing forces are not vertical but oblique. The actual condition is hence one of great complexity, and it may be considered as surprising that the common theory gives results for common cases agreeing so closely with practice. In extreme cases, however, the theory must be supplemented to meet unusual conditions."

Mr. Hudson's paper had for its main object the method of calculating the deflections of plate-web girders as applied to swing or draw bridges, and he found that the result of neglecting to consider the effect of the shearing stresses in producing deflections of the end of a short light span, uniformly loaded, was about 4 per cent.; for the same girder loaded at the extremity the error was 3 per cent. The calculated moment over the central support of a swing bridge due to neglect of the shearing stresses is 2·6 per cent. greater than when these are considered, as well as the stresses resulting from bending. For longer and heavier swing-spans, the consideration of the shearing stresses shows differences about twice as great as for the shorter and lighter structure. It appears therefore to be necessary to consider the effect of the shearing stresses, in modifying the effect of the bending stresses in designing, only for very deep girders subjected to very heavy loads.

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[1] *Trans. Am. Soc. C. E.*, May, 1903.

THE DEFLECTION OF A TRUSS.

The deflection of a truss can be calculated from the general expression—

$$v = \frac{1}{w\mathrm{E}} \Sigma \frac{s\sigma l}{\mathrm{A}}.$$

We may place $w = 1$ ton in the centre of the truss and take $\mathrm{E} = 13{,}400$ tons per sq. in. Then if the truss under consideration is represented by Fig. 252, we may tabulate the quantities represented by the above formulæ as in Table XLVI.

$$\text{Sec. } \theta = 1{\cdot}3. \quad \text{Tan } \theta = \frac{25}{30}. \quad \mathrm{W} = 30 \text{ tons at each bottom apex.}$$

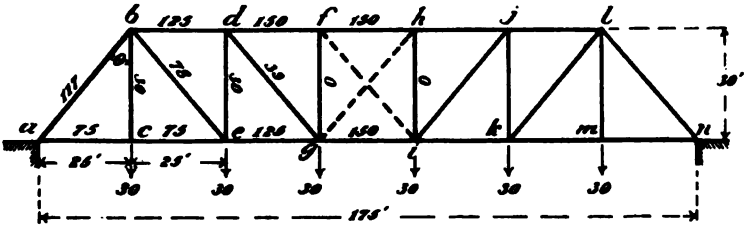


FIG. 252.

The deflection in the centre of the truss due to the deformation of the chords and ends is—

$$\frac{1279{\cdot}4 \times 12}{13400} = 1{\cdot}145 \text{ in.}$$

The stresses in the verticals and diagonals for the unit load w equal to one ton at the centre is shown in Table XLVII., from which it will be observed that there is no stress in fg, hi, bc and lm, due to the unit

TABLE XLVI.

Member.	Stress s.	σ due to 1 ton at the centre.	l in feet.	A Area of cross-section, sq. in.	Value of $\frac{s\sigma l}{A}$
ab	− 117	− 0·65	39	30	98·9
ac	+ 75	$+\frac{5}{12}$	25	17	46·0
ce	+ 75	$+\frac{5}{12}$	25	17	46·0
bd	− 125	$-\frac{5}{6}$	25	32	81·4
eg	+ 125	$+\frac{5}{6}$	25	25	104·2
df	− 150	$-\frac{5}{4}$	25	38	123·4
gi	+ 150	$+\frac{5}{4}$	12·5	30	78·1
fh	− 150	$-\frac{5}{4}$	12·5	38	61·7
					639·7
					2
					1279·4

load at the centre ; hence these bars will not contribute to the deflection. The effect of the verticals and diagonals on the deflection is shown in the following table :—

TABLE XLVII.

Member.	Stress s.	σ due to 1 ton at the centre.	l in feet.	A Area of cross-section, sq. in.	Value of $\frac{s\sigma l}{A}$
bc	+ 30	0	30	6	—
de	− 30	− ·5	30	7·5	60
fg	0	0	—	—	—
be	+ 78	+ ·65	39	16	123·6
dg	+ 39	+ ·65	39	8	123·6
					307·2
					2
					614·4

The deflection in the centre of the truss due to the deformation of the verticals and diagonals is—

$$\frac{614{\cdot}4 \times 12}{13400} = 0{\cdot}55 \text{ in.}$$

The total deflection in the centre is—

$$1{\cdot}145 + 0{\cdot}55 = 1{\cdot}695 \text{ in.} = 1{\cdot}7 \text{ in. say.}$$

If the truss has pin connections there will be a deflection due to the fact that the pins do not fit the holes, which must be added to the elastic deflection.

Let the truss shown in Fig. 253 be loaded at the centre with the unit load w; and the stresses due to this load written on the members

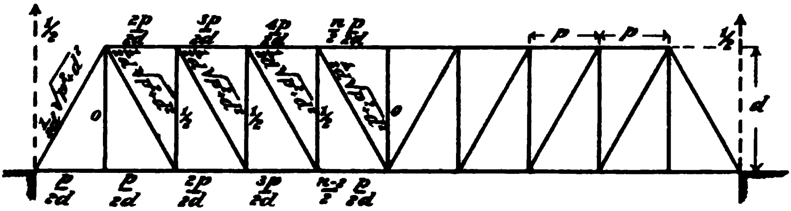


FIG. 253.

as shown. We can find a general expression for the deflection at the centre by finding the values of—

$$vE = S\Sigma\sigma l.$$

For the top chord—

$$S_c\Sigma\sigma l = 2S_c\left\{\frac{2p^2}{2d} + \frac{3}{2}\frac{p^2}{d} + \ldots + \frac{n}{2}\frac{p^2}{2d}\right\} = \frac{p^2S_c}{8d}(n+4)(n-2).$$

For the bottom chord—

$$S_t\Sigma\sigma l = 2S_t\left\{\frac{1}{2}\frac{p^2}{d} + \frac{1}{2}\frac{p^2}{d} + \frac{2p^2}{2d} + \frac{3p^2}{2d} + \ldots + \frac{n-2}{2}\frac{p^2}{2d}\right\}$$

$$= \frac{S_tp^2}{8d}\{n(n-2)+8\}.$$

For the diagonals—

$$S_t\Sigma\sigma l = 2\sqrt{d^2+p^2}\left(\frac{\sqrt{d^2+p^2}}{2d}\right)\left(\frac{n}{2}-1\right)S_t = \frac{S_t}{2d}(d^2+p^2)(n-2).$$

For the verticals—

$$S_c\Sigma\sigma l = \frac{S_cd^2}{2d}(n-4).$$

For the end struts or batter braces—

$$S_c\Sigma\sigma l = \frac{2S_c}{2d}(p^2+d^2)$$

put $S_c = S_t = \frac{S_c+S_t}{2}$, then the value of vE for the whole truss is—

$$vE = \frac{S_c+S_t}{2d}\left\{(n+2)\frac{np^2}{4} + (n-2)d^2\right\}$$

where p and d are in inches.

If we make $n=7$, $p=25'$, $d=30'$, $\frac{S_t+S_c}{2} = 4$, $E = 13400$, $l=175'$, then $v = 1{\cdot}7$ in.

This compares with the foregoing example. Differentiate v with respect to d, to find the effect of varying d, then—

$$\frac{dv}{dd}=\frac{\mathrm{S}_c+\mathrm{S}_t}{2\mathrm{E}}\left\{-\frac{(n+2)np^3}{4d^2}+n-2\right\}=\frac{\mathrm{S}_c+\mathrm{S}_t}{8\mathrm{E}d^2}\{4(n-2)d^2-(n+2)np^2\}$$

put $\frac{dv}{dd}=0$, then for the maximum stiffness—

$$d^2=\frac{p^2}{4}\frac{n+2}{n-2}\,.\,n,$$

$$d=\frac{p}{2}\sqrt{\frac{n+2}{n-2}}\,.\,n.$$

This last expression is derived by Johnson,[1] and shows that for a given amount of material the height of the truss should vary between 1·75 to 2·5 times the panel length; if $n=10$, then $d=1{\cdot}93p$; if $n=7$, $d=1{\cdot}77p$. If the slack in the pin holes averages $\frac{1}{50}$ in., then the inelastic deflection according to Johnson may be expressed as follows:—

$$v'=\tfrac{1}{50}\Sigma\sigma.$$

Assuming that the top member acts as one piece from end to end, and that the verticals are shortened, also the diagonals and bottom chord lengthened at each bay by $\frac{1}{50}$ in., then if the truss has an even number of panels the inelastic deflection,

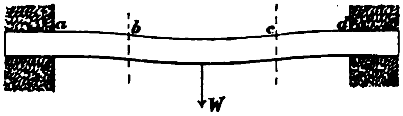
$$v'=\tfrac{1}{50}\Sigma\sigma=\frac{1}{200d}\{(n^2-2n+8)p+(n-4)d+2n\sqrt{p^2+d^2}\}.$$

The camber of the truss should be made equal to the elastic plus the inelastic deflection.

Beams Fixed at the Ends.—A beam is said to be fixed when its ends are loaded in such a manner as to make the tangents at points of support horizontal. The fixing may be effected by building the ends of the beam into a wall (Fig. 254), or by loading the ends as shown in Fig. 255.

When a beam is fixed by either of the above methods there is a downward deflection due to the load, and an upward deflection due to the fixing; hence it follows that there will be two points, *b* and *c*, where the downward deflection changes to an upward deflection; these points are termed "points of inflexion" or of "contra-flexure". The portions from *a* to *b* and from *c* to *d* act as cantilevers, while the portion from *b* to *c* acts as a beam supported at *b*

FIG. 254.



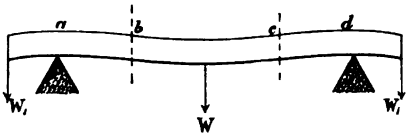


FIG. 255.

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[1] *Framed Structures*, by Johnson.

and *c* and loaded as shown. The upward deflection caused by the upward bending moment is uniform from *a* to *d*. Since the tangents at *a* and *d* are horizontal, it follows that the upward slopes at these points, due to the upward bending moment, must be equal to the downward slope due to the load upon the beam.

Case I.—A beam of uniform section, fixed at both ends and loaded in the centre (Fig. 254).

In the case of a beam supported at both ends, it has been shown that the slope at *a* and *d* is—

$$i_1 = -\frac{Wl^2}{16EI}.$$

If M denote the upward bending moment, which is uniform from *a* to *d*, the slope at *a* and *d* produced by it may be proved to be—

$$i_2 = -\frac{Ml}{2EI},$$ [1]

and since $i_1 + i_2 = 0$—

$$-\frac{Wl^2}{16EI} - \frac{Ml}{2EI} = 0$$

$$\therefore M = -\frac{Wl}{8},$$

which is the bending moment at either *a* or *d*.

If M_1 denote the downward bending moment at any point distant x from *a* or *d*, the total bending moment M_2 at x will be—

$$M + M_1 = M_2$$

$$\therefore M_2 = \frac{Wx}{2} - \frac{Wl}{8}.$$

When $x = \frac{l}{2}$, we obtain the central bending moment, which we will denote by M_3.

$$M_3 = \frac{Wl}{4} - \frac{Wl}{8} = \frac{Wl}{8}.$$

[1] $i = \frac{M}{EI}\int dx = \frac{Mx}{EI} + C$

$\therefore i = 0$ when $x = \frac{l}{2}$

$\therefore C = -\frac{Ml}{2EI}$

$\therefore i = \frac{Mx}{EI} - \frac{Ml}{2EI} = \frac{M}{EI}\left(x - \frac{l}{2}\right).$

The slope at *a* and *d* is found by putting $x = 0$, or $x = l$;

$\therefore i_2 = -\frac{Ml}{2EI}.$

Hence $M = -M_3$, and the bending moments may be represented graphically as in Fig. 256.

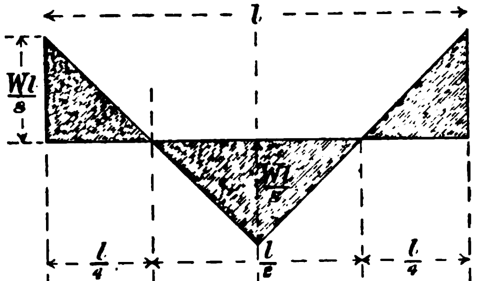


FIG. 256.

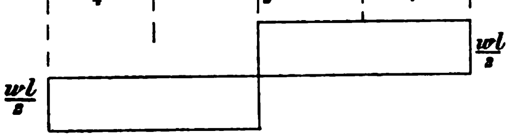


FIG. 257.

To find the points of contra-flexure, we observe that the bending moments must be zero at these points; hence $M_2 = 0$,

$$\therefore \frac{Wx}{2} - \frac{Wl}{8} = 0, \text{ and } x = \frac{l}{4}.$$

This is also clear from the above diagrams, Figs. 256 and 257.

Slope and Deflection under Given Load.—

$$i = \int \frac{M}{EI} dx = \frac{1}{EI} \int \left(\frac{Wx}{2} - \frac{Wl}{8} \right) dx = \frac{Wx^2}{4EI} - \frac{Wlx}{8EI} + C.$$

When $x = 0$, $i = 0$. $\therefore C = 0$.

$$\therefore i = \frac{dv}{dx} = \frac{W}{8EI}(2x^2 - lx)$$

$$v = \frac{W}{8EI}\left(\frac{2x^3}{3} - \frac{lx^2}{2} \right) + C,$$

and $v = 0$ when $x = 0$. $\therefore C = 0$.

$$\therefore v = \frac{W}{8EI}\left(\frac{2x^3}{3} - \frac{lx^2}{2} \right).$$

The maximum slope occurs at the points of contra-flexure, where $x = \frac{l}{4}$; the maximum deflection when $x = \frac{l}{2}$;

$$\therefore i_0 = -\frac{Wl^2}{64EI}$$

$$v_0 = -\frac{Wl^3}{192EI}.$$

Let f denote the maximum intensity of working stress, and W the central breaking load, then—

$$\frac{Wl}{8} = \frac{fI}{y} \quad \therefore W = \frac{8fI}{ly}$$

$$\therefore i_o = -\frac{fl}{8Ey}; \quad v_o = \frac{fl^2}{24Ey}.$$

Case 11.—Beam of uniform section, fixed at the ends and loaded with a uniformly distributed load.

Proceeding in the same manner as before, we obtain the downward slope at the ends, considering the beam as supported—

$$i_1 = -\frac{Wl^2}{24EI}.$$

The upward slope at the ends due to the uniform bending moment M is—

$$i_2 = -\frac{Ml}{2EI}.$$

Therefore, since the slope at the ends must be zero—

$$i_1 + i_2 = 0.$$

$$\therefore -\frac{Wl^2}{24EI} - \frac{Ml}{2EI} = 0; \text{ and } M = -\frac{Wl}{12}.$$

The bending moment at the supports is therefore $-\frac{Wl}{12}$; the downward bending moment at any point distant x from the origin is—

$$M_1 = \frac{W}{2l}(lx - x^2);$$

the total bending moment will therefore be—

$$M_2 = M_1 + M = \frac{W}{2l}(lx - x^2) - \frac{Wl}{12}.$$

This is a maximum when $x = 0$, hence the greatest value of the bending moment is $\frac{Wl}{12}$.

The bending moment at the centre is found by putting $x = \frac{l}{2}$.

$$M_2 = \frac{W}{2l}\left(\frac{l^2}{2} - \frac{l^2}{4}\right) - \frac{Wl}{12} = \frac{Wl}{24}.$$

The bending moments may be represented graphically as in Fig. 258.

To find the points of contra-flexure, we have—

$$M_2 = 0 = \frac{W}{2l}(lx - x^2) - \frac{Wl}{12}$$

$$\therefore x = \frac{l}{2} \pm \frac{l}{\sqrt{12}}$$

which may be written—

$$x = \frac{l}{2} \pm \frac{l}{2\sqrt{3}}$$

as is represented in Fig. 258.

The diagram of shearing stress is shown in Fig. 259.

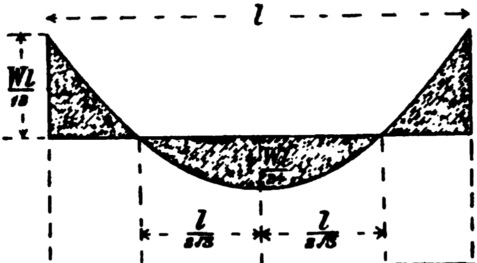


FIG. 258.

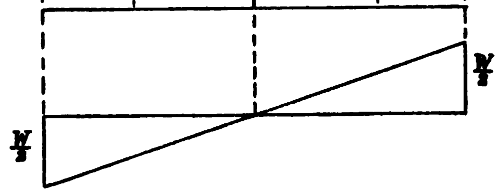


FIG. 259.

Slope and Deflection.—

$$i = \int\frac{M}{EI}dx = \frac{W}{12lEI}(3x^2l - 2x^3 - l^2x),$$

the constant vanishing when $i = 0$ and $x = 0$.

$$v = \frac{W}{12lEI}\left\{lx^3 - \frac{x^4}{2} - \frac{l^2x^2}{2}\right\},$$

the constant vanishing when $v = 0$ and $x = 0$.

Hence for the maximum slope and deflection we have, for slope putting $x = \frac{l}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right)$, and for deflection putting $x = \frac{l}{2}$,

$$i_0 = -\frac{Wl^2}{72\sqrt{3}EI}$$

$$\text{and } v_0 = -\frac{Wl^3}{384EI}.$$

The slope and deflection under the maximum intensity of working stress, denoted by f, may be found thus:—

$$\frac{Wl}{12} = \frac{fI}{y} \quad \therefore W = \frac{12fI}{yl}$$

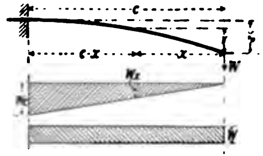
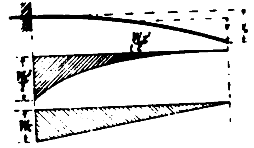
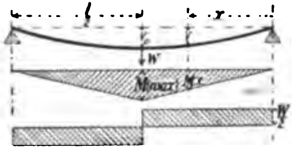
$$\therefore i_0 = -\frac{fl}{6\sqrt{3}Ey}$$

$$v_0 = -\frac{fl^2}{32Ey}.$$

The following table gives the moments, deflections and stresses in beams in a convenient form for reference:—

TABLE XLVIII.

MOMENTS, DEFLECTIONS AND STRESSES IN BEAMS.

Beam with the diagrams of bending moments and shearing stresses.	Mx denotes the equation of bending moments, $M_{max.}$, the maximum moment.	v denotes the equation to the deflection curve, v_o, the maximum deflection.	$\Delta\sigma$ denotes the maximum deflection in terms of the extreme fibre stress σ in symmetrical sections.	σ denotes the maximum intensity of the extreme fibre stress in symmetrical sections.
 FIG. 260.	$Mx = - Wx$ $M_{max.} = - Wc$	$v = \frac{W}{6EI}(2c^3 - 3c^2x + x^3)$ $v_o = - \frac{Wc^3}{3EI}$	$\Delta\sigma = \frac{2\sigma c^2}{3Ed}$	$\sigma = \frac{Wcd}{2I}$
 FIG. 261.	$Mx = - \frac{wx^2}{2}$ $M_{max.} = - \frac{wc^2}{2}$	$v = \frac{w}{24EI}(x^4 - 4c^3x + 3c^4)$ $v_o = \frac{wc^4}{8EI}$	$\Delta\sigma = \frac{\sigma c^2}{2Ed}$	$\sigma = \frac{wc^2d}{4I}$
 FIG. 262.	$Mx = - \frac{Wx}{2}$ $M_{max.} = - \frac{Wl}{4}$	$v = \frac{Wx}{48EI}(3l^2 - 4x^2)$ $v_o = \frac{Wl^3}{48EI}$	$\Delta\sigma = \frac{\sigma l^2}{6Ed}$	$\sigma = \frac{Wld}{8I}$

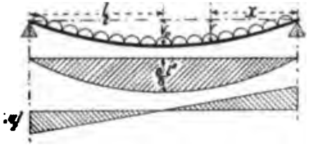
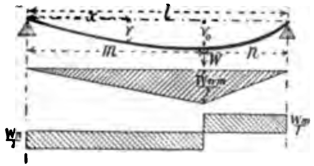
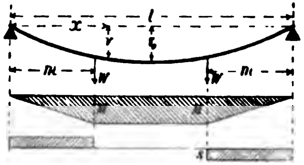
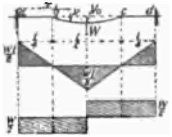
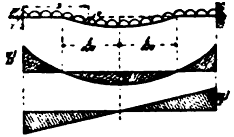
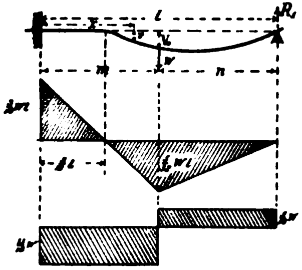
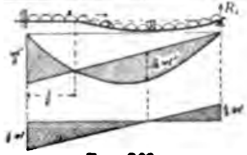
 FIG. 263.	$Mx = \frac{w}{2}(lx - x^2)$ $M_{max.} = \frac{wl^2}{8}$	$v = \frac{w}{24EI}(2lx^3 - x^4 - l^3x)$ $v_o = \frac{5wl^4}{384EI} = \frac{5Wl^3}{384EI}$	$\Delta\sigma = \frac{5\sigma l^2}{24Ed}$	$\sigma = \frac{wl^2d}{16I}$
 FIG. 264.	$x < m$ $Mx = \frac{Wnx}{l}$ $x > m$ $Mx = \frac{Wnx}{l} - W(x - m)$ $M_{max.} = \frac{Wnm}{l}$	$x < m$ $v = \frac{Wnx}{6EIl}(2lm - m^2 - x^2)$ $x > m$ $v = \frac{Wm(l - x)}{6EIl}(2lx -$ $x^2 - m^2)$ $v_o = \frac{Wn}{27EIl}\sqrt{3[m(2n+m)]^3}$ for $x = \frac{1}{3}\sqrt{3[m(2n + m)]}$	$\Delta\sigma = \frac{2\sigma}{27Ed}\sqrt{3m(2n+m)^3}$	$\sigma = \frac{Wmnd}{2Il}$
 FIG. 265.	$x < m$ $Mx = Wx$ $x > m$ $Mx = Wm = M_{max.}$	$x < m$ $v = \frac{Wx}{6EI}(3lm - 3m^2 - x^2)$ $x > m$ $v = \frac{Wm}{6EI}(3lx - 3x^2 - m^2)$ $v_o = \frac{Wm}{6EI}\left(\frac{3l^2}{4} - m^2\right)$	$\Delta\sigma = \frac{\sigma}{3Ed}\left(\frac{3l^2}{4} - m^2\right)$	$\sigma = \frac{Wmd}{2I}$
 FIG. 266.	$Mx = \frac{Wx}{2} - \frac{Wx}{8}$ $M_{max.} = \frac{Wl}{8}$	$v = \frac{W}{8EI}\left(\frac{2x^3}{3} - \frac{lx^2}{2}\right)$ $v_o = \frac{Wl^3}{192EI}$	$\Delta\sigma = \frac{\sigma l^2}{12Ed}$	$\sigma = \frac{Wld}{16I}$

TABLE XLVIII.—*continued.*

Beam with the diagrams of bending moments and shearing stresses.	Mx denotes the equation of bending moments, $M_{max.}$, the maximum moment.	v denotes the equation to the deflection curve, v_o, the maximum deflection.	$\Delta\sigma$ denotes the maximum deflection in terms of the extreme fibre stress σ.	σ denotes the maximum intensity of the extreme fibre stress in symmetrical sections.
 FIG. 267.	$Mx = \frac{Wx}{2l}(l - x) - \frac{Wl}{12}$ $M_{max.} = \frac{Wl}{12} = \frac{wl^2}{12}$	$v = \frac{W}{12EIl}\left(lx^3 - \frac{x^4}{2} - \frac{l^2x^2}{2}\right)$ $\bar{v}_o = \frac{Wl^3}{384EI} = \frac{wl^4}{384EI}$	$\Delta\sigma = \frac{\sigma l^2}{16Ed}$	$\sigma = \frac{Wld}{24I}$
 FIG. 268.	$R_1 = \frac{W}{2l^3}(3lm^2 - m^3)$ $x < m$ $Mx = R_1(l - x) - W(m - x)$ $x > m$ $Mx = R_1(l - x)$; $M_{max.} = R_1(l - m)$ If $m = n = \frac{l}{2}$, $R = 5/16W$, $M = 5/32Wl$	$x < m$ $v = \frac{1}{6EI}(R_1x^3 - 3R_1lx^2 +$ $3Wmx^2 - Wx^3)$ $x > m$ $v = \frac{1}{6EI}(R_1x^3 - 3R_1lx^2 +$ $3Wm^2x - Wm^3)$ $v_o = \frac{Wm^2}{6EI}(l - m)\sqrt{\frac{l-m}{3l-m}}$ for $x = l\left(1 - \sqrt{\frac{l-m}{3l-m}}\right)$	$\Delta\sigma = \frac{2\sigma}{3Ed}\sqrt{(3l-m)(l-m)}$	$\sigma = \frac{Wd}{4l^3I}(3lm^2 - m^3)(l - x)$
 FIG. 269.	$R_1 = 3/8wl$ $Mx = \frac{w}{8}(4x - l)(l - x)$ $M_{max.} = -\frac{wl^2}{8}$, for $x = 0$	$v = \frac{wx^2}{48EI}(l - x)(3l - 2x)$ $v_o = 0{\cdot}0054\frac{wl^4}{EI}$ for $x = 0{\cdot}578l$	$\Delta\sigma = \frac{0{\cdot}0864\sigma l^2}{Ed}$	$\sigma = \frac{wl^2d}{16I}$

CHAPTER XI.

CONTINUOUS GIRDERS AND THEIR APPLICATION IN ROAD AND RAILWAY BRIDGES—BRESSUMMER BEAMS AND WAREHOUSE FLOORS.

A GIRDER supported at each end and at one or more intermediate piers is said to be continuous over these piers. In the case of a girder 150 ft. long, supported at each end and at two intermediate piers spaced 50 ft. centres, the girder is said to be continuous over two piers in spans of 50 ft.

The complete investigation of the stresses in continuous girders, and the slope and deflections, will not be attempted in this work, but the results of these investigations will be made use of sufficiently to illustrate the methods of designing continuous girders for various purposes.

Two cases will be considered in the first instance :—

1. When the load on any span is uniformly distributed over that span.

2. When the loads are concentrated at one or more points in a span.

A third case might occur with both distributed and concentrated loads, but this could obviously be treated by combining the results of each loading considered separately.

The assumptions made in the various examples considered in this chapter are :—

1. That all the supports are on the same level or on the same uniform gradient.

2. That the girder is uniform in section throughout its length.

With regard to the second assumption, which is never realised in bridges, in which the material is disposed in accordance with the variations in the stresses, the errors in the stresses need not lead to errors in the design of the various members of any practical importance.

Let M_0, M_1, M_2 and M_3, Fig. 271, denote the bending moments at the points of support A, B, C and D respectively.

Let l_1, l_2 and l_3 be the lengths of the spans, and w_1, w_2 and w_3 denote the loads per lineal foot.

Then, on the assumptions stated, it can be proved that—

$$M_0 l_1 + 2M_1(l_1 + l_2) + M_2 l_2 + \tfrac{1}{4}(w_2 l_2^3 + w_1 l_1^3) = 0.$$

This equation expresses the relation between the bending moments at any three consecutive piers; it is known as the equation of three moments, and was first demonstrated by Clapeyron.

The moments M_1 and M_2 in Fig. 271 tend to bend the girders upwards; thus in Fig. 270, which shows a continuous girder of three spans, the deflection over the piers B and C is upwards, and the

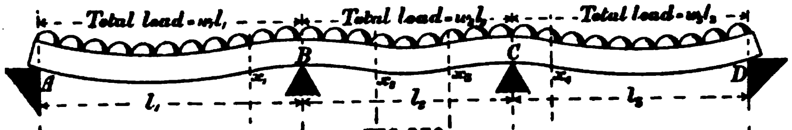


FIG. 270.

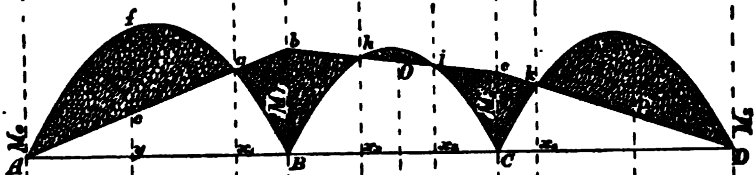


FIG. 271.

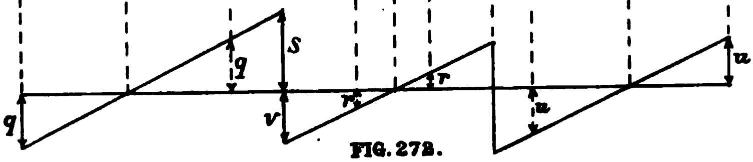


FIG. 272.

diagram of upward bending moments is represented by the figure A*bc*D, Fig. 271. The bending moments over the points A and D are zero.

Beside the upward bending moments, which attain their maximum value for adjacent spans at the points B and C, there are downward bending moments between the supports, which act in precisely the same manner as in ordinary beams supported at AB, BC and CD. The resultant bending moment at any point is the difference between the upward and downward bending moments at that point.

Let the curves described upon AB, BC and CD denote the bending moments over these spans respectively considered as detached beams. In the case of uniform loading these curves are parabolas, and since the upward bending moments are denoted by the figure A*bc*D, the shaded areas represent the resultant bending moment; thus the bending moment at the point y is denoted by the ordinate ef.

At the points g, h, j and k, corresponding with the points x_1, x_2, x_3 and x_4 on the girder, the bending moment is zero, or the upward bending moment is exactly balanced by the downward bending moment. The points x_1, x_2, x_3 and x_4 are termed points of inflection or contra-flexure. The portion of the span AB between A and x_1 is subjected to the same bending moments as an ordinary girder supported at A and x_1 and loaded with the same load per foot run as that on the span AB. The portions between x_2 and x_3 in the span BC, also between x_4 and D in the span CD, are similarly subjected to the same bending moments as in ordinary beams supported at points corresponding with the points of contra-flexure, x_2, x_3 and x_4, and loaded with the same load per foot run as that on the spans BC and CD respectively.

The portions x_1B, Bx_2, x_3C and Cx_4 act as cantilevers, each loaded with a uniform load corresponding with that on the particular span in which the portion referred to occurs, in addition to a load at their extremities corresponding with the reactions at x_1, x_2, x_3 and x_4 considered as ordinary detached girders supported at Ax_1, x_2x_3, and x_4D respectively.

The shearing stresses may therefore be derived by considering the whole girder between A and D as made up of detached girders and cantilevers in the manner described, thus: The stearing stresses q, q, u and u are each half the load on the spans Ax_1 and x_4D; the portions rr are the shearing stresses at x_2 and x_3, and are equal to half the load on the portion x_2x_3.

The shearing stresses on the girder are represented by the diagram, Fig. 272.

The pressures on the supports A and D are equal to the shearing stresses at these points, and the pressures at B and C to the sum of the shearing stresses from the adjacent spans; thus at B the pressure is $s + v$.

The method of calculating the stresses in continuous girders may be summarised as follows:—

1. Find the bending moments over the piers, and draw the diagram of upward bending moments A$bc$D, Fig. 271.

2. Draw the diagram of downward bending moments on the same datum line, AD, Fig. 271, and on the same side of it as the diagram of upward bending moments; then find the points of contra-flexure and the resultant bending moments in each span.

3. Draw the shearing-stress diagrams for each span, Fig. 272, and obtain the pressures on the piers.

When the bending moments over the piers are obtained, the equations of bending moments and shearing stresses for each span can be easily written down and the results tabulated.

Example.—A railway bridge for a single line is constructed with two main girders, each 480 ft. long, continuous over two piers, forming three spans each 159 ft. centres.

The effective depth measured between the centres of gravity of the top and bottom chords, or booms, is 12 ft.

The dead load on each main girder is 0·6 ton per foot run.

The equivalent uniformly distributed load is 0·7 ton per foot run.

The total load on a span is therefore 1·3 ton per foot run.

In the equation of three moments $l_1 = l_2 = l_3$ and $M_0 = M_3 = 0$.

Case I.—The first span loaded with the live and dead load ($w_1 = 1\cdot3$), the remaining spans loaded with the dead load only ($w_2 = w_3 = 0\cdot6$).

Then substituting in the three-moment equation, we obtain—

$$\left.\begin{aligned} 2M_1(2l) + M_2 l &= -\tfrac{1}{4}(w_2 l_2^3 + w_1 l_1^3) \\ M_1(l) + 2M_2(2l) &= -\tfrac{1}{4}(w_3 l_3^3 + w_2 l_2^3) \end{aligned}\right\}$$

$$\begin{aligned} \therefore\ 16M_1 + 4M_2 &= -0\cdot6 \times \overline{159}^2 - 1\cdot3 \times \overline{159}^2 = -\overline{159}^2(1\cdot9) \\ 4M_1 + 16M_2 &= -2 \times 0\cdot6 \times \overline{159}^2 = -\overline{159}^2(1\cdot2) \\ \therefore\ M_1 &= -2696\cdot64 \text{ foot-tons} \\ \text{and } M_2 &= -1222 \text{ foot-tons.} \end{aligned}$$

The equation of bending moments for the first span is—

$$y = 0\cdot65(159x - x^2) - 16\cdot96x.$$

To find the point of contra-flexure, let $y = 0$, then $x = 132\cdot9$ ft.; hence the point of contra-flexure is 26·1 ft. from the first pier.

The equation of bending moments in the middle span is—

$$y = 0\cdot3(159x - x^2) - 1222 - 9\cdot27x,$$

where x is measured from the second pier. When $y = 0$, $x = 69\cdot4$ and 58·7 ft.; hence the points of contra-flexure occur 89·6 ft. from the first pier, and 58·7 ft. from the second pier.

It will not be necessary to consider the third span, as, when the live load approaches from the right instead of the left, the stresses will be the same as in the first span already considered, which will exceed considerably the stresses due to the dead load only on this span.

Shearing Stresses.—

$$\text{First span at abutment} = \frac{132\cdot9 \times 1\cdot3}{2} = 86\cdot38 \text{ tons.}$$

$$\text{,, ,, first pier} = (66\cdot45 + 26\cdot1)\,1\cdot3 = 120\cdot31 \text{ tons.}$$

$$\text{Second span at first pier} = 94\cdot95 \times 0\cdot6 = 57\cdot0 \text{ tons.}$$

$$\text{,, ,, at second pier} = 64 \times 0\cdot6 = 38\cdot4 \text{ tons.}$$

The diagram of bending moments and shearing stresses is shown in Figs. 273 and 274.

If the live load had extended from the middle of the first span to the middle of the second span, the moment over the first pier would have been about $1\frac{1}{2}$ per cent. greater than that found in Case I., but partially distributed live loads will not be considered. If the train approaches from the right headed by two engines, the whole bridge may be covered with the live load; this may be taken as uniform throughout.

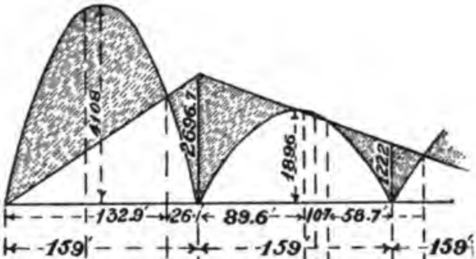


FIG. 273.

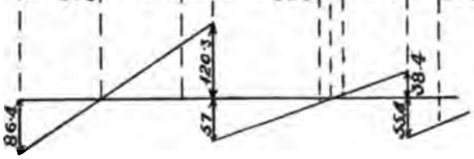


FIG. 274.

Case II.—Three spans loaded with 1·3 ton per foot run.

The three-moment equation becomes—

$$2M_1(2l) + M_2 l = -\tfrac{1}{4}(2w_1 l^3)$$

$$\left.\begin{aligned} \therefore 8M_1 + 2M_2 &= -w_1 l^2 \\ \text{also } 8M_2 + 2M_1 &= -w_1 l^2 \end{aligned}\right\}$$

$$\text{hence } M_1 = M_2 = -\frac{w_1 l^2}{10} = -\frac{1\cdot3 \times \overline{159}^2}{10} = -3286\cdot5 \text{ foot-tons.}$$

The equation of bending moments for the first and third spans becomes—

$$y = 0\cdot65(159x - x^2) - 20\cdot67x$$

where x = the distance from the left and right abutment.

$$\text{If } y = 0,\ x = 127\cdot2 \text{ ft.}$$

The equation of bending moments in the middle span is—

$$y = -3286\cdot5 + 0\cdot65(159x - x^2)$$

from which we find, by making $y = 0$, $x = 115$ and 44 ft.; hence the points of contra-flexure occur 44 ft. from each pier.

Shearing Stresses.—

$$\text{First and third spans over abutments} = \frac{127\cdot2 \times 1\cdot3}{2} = 82\cdot68 \text{ tons.}$$

$$\text{,, \quad ,, \quad over piers} = 95\cdot4 \times 1\cdot3 = 124\cdot02 \text{ ,,}$$

$$\text{Middle span over piers} = \frac{159 \times 1\cdot3}{2} = 103\cdot35 \text{ tons.}$$

Comparing Cases I. and II., we observe that the shearing stresses and bending moments over the piers is greater in Case II., while the bending moments between the left abutment and point of contra-flexure is greater in Case I.

Case III.—First and second spans loaded with 1·3 ton per foot run; third span loaded with 0·6 ton per foot run.

$$\left.\begin{aligned} \text{Here } 8M_1 + 2M_2 &= -w_1 l^2 \\ 16M_2 + 4M_1 &= -(w_2 l^2 + w_3 l^2) \end{aligned}\right\} \text{where } w_1 = w_2$$

$$\therefore M_1 = -(7w_1 - w_3)\frac{l^2}{60} = -(9{\cdot}1 - 0{\cdot}6)\frac{25281}{60} = -3581{\cdot}3 \text{ foot-tons.}$$

$M_2 = -2106{\cdot}7$ foot-tons.

Equation to first span is—

$$y = 0{\cdot}65(159x - x^2) - 22{\cdot}52x$$

therefore the point of contra-flexure is $x = 124{\cdot}35$.

Equation to second span is—

$$y = 0{\cdot}65(159x - x^2) - 2106{\cdot}7 - 9{\cdot}27x,$$

therefore the points of contra-flexure occur at 42·65 ft. from the first pier, and 28·35 ft. from the second pier.

Case IV.—First and third spans loaded with 0·6 ton per foot run; middle span loaded with 1·3 ton per foot run.

$$\left.\begin{aligned} \text{Here } 16M_1 + 4M_2 &= -(w_1 l^2 + w_2 l^2) \\ 16M_2 + 4M_1 &= -(w_2 l^2 + w_1 l^2) \end{aligned}\right\}$$

$$\therefore M_1 = M_2 = -2401{\cdot}7 \text{ foot-tons.}$$

Only the middle span need be considered for this case, the equation of which is—

$$y = 0{\cdot}65(159x - x^2) - 2401{\cdot}7,$$

therefore the points of contra-flexure occur at 28·27 ft. from the piers.

The shearing stress over the piers for the middle span is the same as in Case II.

Case V.—First and third spans loaded with 1·3 ton per foot run; middle span loaded with 0·6 ton per foot run.

Here $M_1 = M_2$, as in Case IV., *viz.*, 2401·7 foot-tons.

Equation of first and third spans is—

$$y = 0{\cdot}65(159x - x^2) - 15{\cdot}105x,$$

therefore the points of contra-flexure occur at $x = 135{\cdot}76$ ft. from either abutment.

The shearing stresses are less than in the cases already considered.

Comparing the foregoing cases, we see that the maximum bending moment for the first and third spans between the point of contra-flexure and the abutments occurs for the loading considered under Case V., which gives the longest effective span.

The maximum moment over the piers and in the cantilever portions occurs for the loading considered in Case III. The middle span is subjected to greatest bending moment about the centre in Case IV., and in the cantilever portions in Cases I., III. and V.

The maximum shearing stresses in the side spans occur in Cases III. and V., and in the middle span in Cases III. and IV.

The diagram of shearing stresses for a moving load will be semiparabolas between the abutments and points of contra-flexure in the first and third spans, and straight lines tangential to the parabolas for the cantilever portions. In the central span the shearing stresses in the portion between the two points of contra-flexure will be represented by parabolas and the two cantilevers on each side by straight lines.

The diagram of maxima bending moments is shown in Fig. 275.

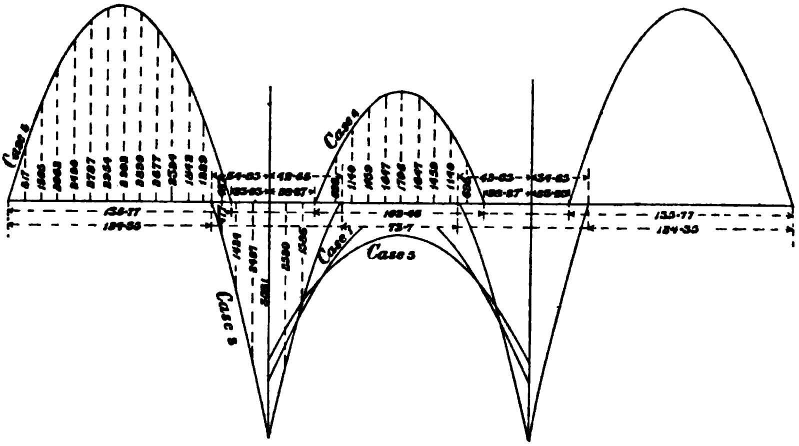


FIG. 275.

The moments of resistance of the flange plates provided to resist them may be plotted.

The diagram of maxima shearing stresses is shown in Fig. 276.

It will be observed that the points of contra-flexure are continually changing during the passage of the live load, hence the working stress must be taken much lower about the region of the points of contra-flexure, and the diagonal members of the web must be counterbraced for reversals of stress, or designed for compression as well as tension.

Concentrated Loads.—The equation of three moments for concentrated loads, for the case when all the supports are on the same level or on the same uniform gradient, and the section of the girder is uniform throughout its length, may be expressed as follows :—

$$\frac{M_2}{3}(l_1 + l_2) + \frac{M_1 l_1}{6} + \frac{M_3 l_2}{6} + \Sigma\frac{W'x'}{6l_2}(l_2^2 - x'^2) + \Sigma\frac{Wx}{6l_1}(l_1^2 - x^2) = 0,$$

where—

$$\Sigma\frac{Wx}{6l_1}(l_1^2 - x^2) = \frac{W_1 x_1}{6l_1}(l_1^2 - x_1^2) + \frac{W_2 x_2}{6l_1}(l_1^2 - x_2^2) + \ldots$$

$$\Sigma\frac{W'x'}{6l_2}(l_2^2 - x_1^2) = \frac{W_1' x_1'}{6l_2}(l_2^2 - x_1'^2) + \frac{W_2' x_2'}{6l_2}(l_2^2 - x_2'^2) + \ldots$$

The use of this equation will be illustrated in the following example. A bressummer beam of two spans, continuous over one support,

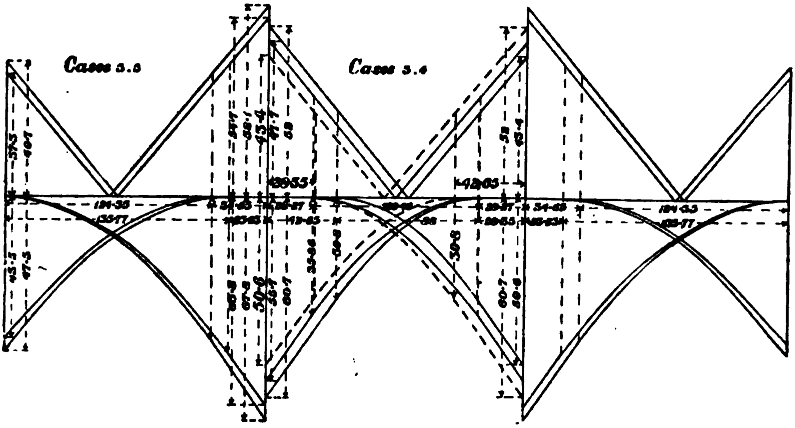


Fig. 276.

carries a shop-front and portions of the floors above. The whole of the load is applied to the beam through masonry piers built upon the

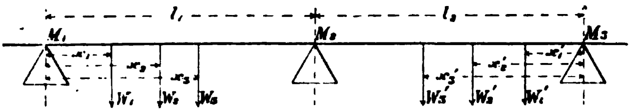


Fig. 277.

upper flange. The magnitudes and points of application of these loads are illustrated in Figs. 277 and 278. In this case $M_1 = M_3 = 0$, and the equation becomes—

$$\frac{23\cdot1}{.3}M_2 + \frac{67 \times 10}{6 \times 12\cdot5}(\overline{12\cdot5}^2 - 10^2) + \frac{30}{6 \times 12\cdot5}(\overline{12\cdot5}^2 - 1^2)$$

$$+ \frac{11\cdot5 \times 9}{6 \times 10\cdot6}(\overline{10\cdot6}^2 - 9^2) + \frac{11\cdot5 \times 4\cdot5}{6 \times 10\cdot6}(\overline{10\cdot6}^2 - \overline{4\cdot5}^2)$$

$$+ \frac{60 \times 2\cdot2}{6 \times 10\cdot6}(\overline{10\cdot6}^2 - \overline{2\cdot2}^2) = 0$$

$$\therefore\ 7\cdot7M_2 + 502\cdot5 + 62\cdot1 + 51\cdot03 + 74\cdot94 + 223\cdot1 = 0$$

$$\therefore\ 7\cdot7M_2 = -913\cdot67$$

$$\therefore\ M_2 = -118\cdot6 \text{ foot-tons.}$$

The diagram of bending moments A*bcd*B and B*fg*C, for the spans AB and BC considered as detached girders, should be plotted to a sufficiently large scale, as well as the diagram of upward bending moments APC, and their intersections noted, from which the points of contra-flexure x_1 and x_2 are obtained, 3·7 ft. from B in the first span, and 1·8 ft. from B in the second span.

The shearing-stress diagram is drawn by considering Ax_1 and Cx_2 as detached spans. The cantilever x_1B is loaded at its extremity with the shearing stress at x_1; it is also loaded with 11·5 tons concentrated at a point 1·6 ft. from B. The cantilever Bx_2 is loaded at its extremity with 65·4 tons.

The diagrams of bending moments and shearing stresses are illustrated in Figs. 278 and 279.

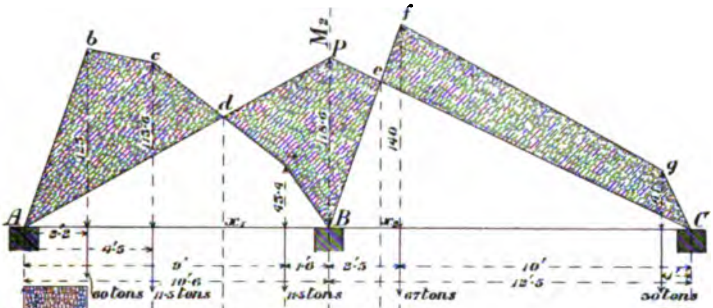


FIG. 278.

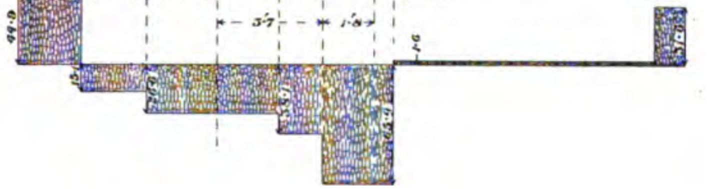


FIG. 279.

The load upon the central support is—

$$65{\cdot}4 + 38{\cdot}1 = 103{\cdot}5 \text{ tons.}$$

The maximum bending moment occurs at B, *viz.*, 118·6 foot-tons.

The beam is constructed of two Dorman rolled steel girders 12 in. deep, with flanges 6 in. × 1 in., and webs $\frac{3}{4}$ in. thick. The flanges are connected together on the top by means of a plate running the whole length. The web is stiffened by means of channel steel stiffeners. The working stress is 9 tons per sq. in., the tensile strength of the steel being 32 tons per sq. in.

In consequence of the change in the position of the points of contra-flexure during the passage of a rolling load, the shearing stresses cannot be accurately determined without considerable labour.

In the first example, when the uniform live load of 0·7 ton per foot run advances from one abutment and gradually covers the three spans, the point of contra-flexure in the first span moves through a distance of 135·8 − 124·3 = 11·5 ft.

If we draw the diagrams of shearing stresses for the extreme positions of the point of contra-flexure, we see that the nearer the point is to the pier the smaller the stress on the pier and the greater the stress on the abutment, so that if we use the point nearer the pier for determining the shearing stresses on the half of the effective span nearest the abutment, and the point farther from the pier for determining the shearing stresses for the remainder of the effective span and cantilever portions, we shall obtain stresses which do not differ much from the true maxima. We can deal with the central span in a similar manner. If the girder is of the lattice type, we first find the extreme positions of the points of contra-flexure, then apply the methods explained in Chapter VIII.

Concentrated rolling loads may be dealt with by calculating the bending moments and shearing stresses for various positions of the wheel loads in the manner illustrated in the foregoing example of the bressummer beam. The positions of the wheel loads which produce maxima stresses could, of course, be found if the points of contra-flexure are known; but this is not the most convenient method, excepting in the case of a traction engine with only two axles, as shown in Fig. 390. In this case the points of contra-flexure for the dead load will not be moved very much with the traction engine in any position, so that, if we make two calculations for the extreme positions of the point of contra-flexure, we generally cover the maxima stresses in this case also.

Economy of the Continuous Girder.—In a three-span bridge of the best proportions, *i.e.*, in which the effective spans are about equal in length under the loads which produce maxima stresses, or in which the side spans are about four-fifths of the central span, the economy of material due to continuity is about 50 per cent. for the dead load, and 16 per cent. for the live load, over three independent spans; hence the advantage of continuous girder road bridges where the dead load is considerable. This advantage will vanish if the piers settle unequally.

Influence Lines applied to Continuous Girders.—We may investigate the stresses in continuous girders by means of influence lines in the following manner, Fig. 280. The equation of three moments for any system of loading may be written thus :—

$$M_2 l_2 + 2M_3(l_2 + l_3) + M_4 l_3 = -P_2 l_2^2(k - k^3) - P_3 l_3^2(2k - 3k^2 + k^3).$$

The proof of this theorem is given in Lanza's *Applied Mechanics*, and in Merriman and Jacoby's *Higher Structures*, Part IV. It is not necessary to give the details of the investigations here, or to consider the effect of the yielding of the piers.

In the cases considered, k is the distance per unit of length from the nearest support to the left. The last term of the general equation, above stated, is obtained from the last but one, by putting $(1 - k)$ for k as is apparent from the figure, so that for our purpose we may consider only the equation in the form—

$$M_2 l_2 + 2M_3(l_2 + l_3) + M_4 l_3 = -P_2 l^2 K.$$

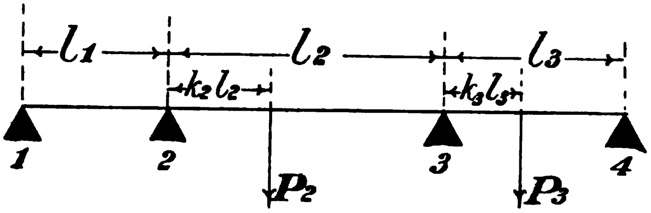


Fig. 280.

Where K_1 is written for $(k - k^3)$ for the sake of brevity, also K_2 is written for $\{(1 - k) - (1 - k)^3\}$ for the same reason. For the purpose of getting influence lines P_2 is taken as equal to unity. The following table by Hawken[1] gives the values of $(k - k_3) = K_1$ and of $(2k - 3k^2 + k^3) = K_2$.

TABLE XLIX.

Read down for K_1.

k	0	1	2	3	4	5	6	7	8	9		
·0	·0000	·0100	·0200	·0300	·0399	·0499	·0598	·0697	·0795	·0893	·0990	·9
·1	·0990	·1087	·1183	·1278	·1373	·1466	·1559	·1651	·1742	·1831	·1920	·8
·2	·1920	·2007	·2094	·2178	·2262	·2344	·2424	·2503	·2580	·2656	·2730	·7
·3	·2730	·2802	·2872	·2941	·3007	·3071	·3133	·3193	·3251	·3307	·3360	·6
·4	·3360	·3411	·3459	·3505	·3548	·3589	·3627	·3662	·3694	·3724	·3750	·5
·5	·3750	·3773	·3794	·3811	·3825	·3836	·3844	·3848	·3849	·3846	·3840	·4
·6	·3840	·3830	·3817	·3800	·3779	·3754	·3725	·3692	·3656	·3615	·3570	·3
·7	·3570	·3521	·3468	·3410	·3348	·3281	·3210	·3135	·3054	·2970	·2880	·2
·8	·2880	·2786	·2686	·2582	·2473	·2359	·2239	·2115	·1985	·1850	·1710	·1
·9	·1710	·1564	·1413	·1256	·1094	·0926	·0753	·0573	·0388	·0197	·0000	·0
		9	8	7	6	5	4	3	2	1	0	k

Read up for K_2.

Continuous Girder. Two equal spans and three supports.—Fig. 281.

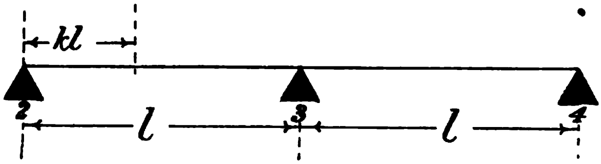


Fig. 281.

(*a*) Bending moment at 3—

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[1] *Proc. Sydney University Engineering Society*, 1903.

Since $l_1 = l_2 = l$ say.

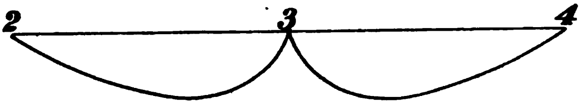
$M_2 l + 4M_3 l + M_4 l = -l^2(K_1)$, for load on span 2-3

$\therefore M_2 + 4M_3 + M_4 = -lK_1$.

$M_2 = M_4 = 0$

$\therefore 4M_3 = -lK_1$ and $M_3 = -\frac{l}{4}K_1 = -\frac{l}{4}(k - k^3)$.

This is a maximum when $1 - 3k^2 = 0$; *i.e.*, $k = \frac{1}{\sqrt{3}}$.



Bending Moment at 3.
Fig. 282.

The figures tabulated are M_3/l.

The following table gives the bending moments at 3.

[1] $\int_{k=0}^{k=1} M_3 = -\frac{1}{16}Wl.$

TABLE L.

k	0	1	2	3	4	5	6	7	8	9
·0	·0000	·0025	·0050	·0075	·0100	·0125	·0150	·0174	·0199	·0223
·1	·0247	·0272	·0296	·0319	·0343	·0367	·0390	·0413	·0436	·0458
·2	·0480	·0502	·0524	·0545	·0566	·0586	·0606	·0626	·0645	·0664
·3	·0683	·0701	·0718	·0735	·0752	·0768	·0783	·0798	·0813	·0827
·4	·0840	·0853	·0865	·0876	·0887	·0897	·0907	·0916	·0924	·0931
·5	·0938	·0943	·0949	·0953	·0956	·0959	·0961	·0962	·0962	·0962
·6	·0960	·0958	·0954	·0950	·0945	·0939	·0931	·0923	·0914	·0904
·7	·0893	·0880	·0867	·0853	·0837	·0820	·0803	·0784	·0764	·0743
·8	·0720	·0697	·0672	·0646	·0618	·0590	·0560	·0529	·0496	·0463
·9	·0428	·0391	·0353	·0314	·0274	·0232	·0188	·0143	·0097	·0049

The maximum value of M_3 occurs when $k = \frac{1}{\sqrt{3}}$, $M_3 = ·0962l$.

When the load is on the span 3-4, $M_3 = -\frac{1}{4}lK_2$; for arithmetical values, see Table LI.(β).

(*b*) Reactions—

Take reactions upwards ↑ + positive, and downwards ↓ − negative.

To find the reactions R_2, R_3 and R_4: Consider the span 2-3 loaded, and take moments about the point 3—

$$R_2 = \frac{M_3}{l} + (1 - k). \quad R_4 = \frac{M_3}{l}.$$

---

[1] $\int_{k=0}^{k=1} M_3$ is written for $\int_0^1 M_3 dk$.

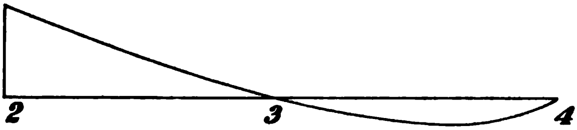
Take moments about the point 4—

$$R_3 = \frac{1}{l}[-R_2 \times 2l + (2l - kl)]$$

$$= -2R_2 + (2 - k) = -\frac{2M_3}{l} + k.$$

Consider the span 3-4 loaded: $R_2 = R_4$ (of above) with $(1 - k)$ put for k, and R_3 is found by symmetry by putting $(1 - k)$ for k in the above equation.

Influence line of R_2 (Fig. 283)—



Reaction at 2.
Fig. 283.

Load on span 2-3. $R_2 = \frac{M_3}{l} + (1 - k) = -\frac{1}{4}(k - k^3) + (1 - k);$

$$\therefore \int_{k=0}^{k=1} R_2 = +7/16W.$$

The maximum or minimum value occurs when $\frac{dR_2}{dk} = 0$, when $-\frac{1}{4}(1 - 3k^2) - 1 = 0$; *i.e.*, $k = \pm\sqrt{\frac{5}{3}}$.

Load on span 3-4—

$$R_2 = \frac{M_3}{l} = -\frac{1}{4}K_2;$$

$$\therefore \int_{k=0}^{k=1} R_2 = -\frac{1}{16}W.$$

Maximum or minimum when $k = 1 \pm \frac{1}{\sqrt{3}}$. R_4 is the same as R_2 when reckoned from the other end support.

Table LI.

Reactions at 2.

(*a*) Load on 2-3. $R_2 = -\frac{1}{4}K_1 + (1 - k)$. All figures positive.

k	0	1	2	3	4	5	6	7	8	9
·0	+1·0000	·9875	·9750	·9625	·9500	·9375	·9250	·9126	·9001	·8877
·1	+·8753	·8628	·8504	·8381	·8257	·8133	·8010	·7887	·7764	·7642
·2	+·7520	·7398	·7276	·7155	·7034	·6914	·6794	·6674	·6555	·6436
·3	+·6317	·6199	·6082	·5965	·5848	·5732	·5617	·5502	·5387	·5273
·4	+·5160	·5047	·4935	·4824	·4713	·4603	·4493	·4384	·4276	·4169
·5	+·4062	·3957	·3851	·3747	·3644	·3541	·3439	·3338	·3238	·3138
·6	+·3040	·2942	·2846	·2750	·2655	·2561	·2469	·2377	·2286	·2196
·7	+·2107	·2020	·1933	·1847	·1763	·1680	·1597	·1516	·1436	·1357
·8	+·1280	·1203	·1128	·1054	·0982	·0910	·0840	·0771	·0704	·0637
·9	+·0572	·0509	·0447	·0386	·0326	·0268	·0212	·0157	·0103	·0051

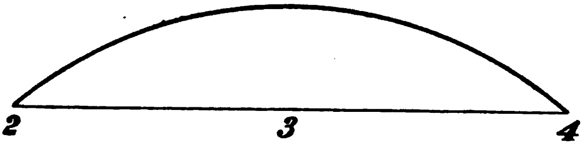
TABLE LI. (*continued*).

(β) Load on 3-4. $R_2 = -\frac{1}{4} K_2$. All figures negative.

k	0	1	2	3	4	5	6	7	8	9
·0	·0000	·0049	·0097	·0143	·0188	·0232	·0274	·0314	·0353	·0391
·1	·0428	·0463	·0496	·0529	·0560	·0590	·0618	·0646	·0672	·0697
·2	·0720	·0743	·0764	·0784	·0803	·0820	·0837	·0853	·0867	·0880
·3	·0893	·0904	·0914	·0923	·0931	·0939	·0945	·0950	·0954	·0958
·4	·0960	·0962	·0962	·0962	·0961	·0959	·0956	·0953	·0949	·0943
·5	·0938	·0931	·0924	·0916	·0907	·0897	·0887	·0876	·0865	·0853
·6	·0840	·0827	·0813	·0798	·0783	·0768	·0752	·0735	·0718	·0701
·7	·0683	·0664	·0645	·0626	·0606	·0586	·0566	·0545	·0524	·0502
·8	·0480	·0458	·0436	·0413	·0390	·0367	·0343	·0319	·0296	·0272
·9	·0247	·0223	·0199	·0174	·0150	·0125	·0100	·0075	·0050	·0025

Maximum when $k = 1 - \frac{1}{\sqrt{3}}$, $R_4 = -\cdot 0962$.

Influence line of R_3 (Fig. 284)—



Reaction at 3.
FIG. 284.

Load on 2-3. $R_3 = -\frac{2M_3}{l} + k = \frac{1}{2}(3k - k^3)$.

$\int_{k=0}^{k=1} R_3 = \frac{5}{8}W$. Maximum occurs when $3 = 3k^2$; *i.e.*, $k = \pm 1$.

TABLE LII.

REACTIONS AT 3.

k	0	1	2	3	4	5	6	7	8	9
·0	·0000	·0150	·0300	·0450	·0600	·0750	·0899	·0949	·1198	·1347
·1	·1495	·1644	·1792	·1939	·2087	·2233	·2380	·2526	·2671	·2816
·2	·2960	·3104	·3247	·3389	·3531	·3672	·3812	·3952	·4090	·4228
·3	·4365	·4501	·4636	·4771	·4904	·5036	·5167	·5297	·5426	·5554
·4	·5680	·5806	·5930	·6053	·6174	·6295	·6414	·6531	·6647	·6762
·5	·6875	·6987	·7097	·7206	·7313	·7418	·7522	·7624	·7725	·7823
·6	·7920	·8015	·8109	·8200	·8290	·8377	·8463	·8546	·8628	·8708
·7	·8785	·8861	·8934	·9005	·9074	·9141	·9205	·9268	·9327	·9385
·8	·9440	·9493	·9543	·9591	·9637	·9680	·9720	·9758	·9793	·9825
·9	·9855	·9882	·9907	·9928	·9947	·9963	·9977	·9987	·9994	·9999

Load on 3-4.

R_3 by symmetry has the same values if we measure distances from the centre support 3, *i.e.*, if we read from the right end for k instead of the left end. By referring to Fig. 282 it will be seen that the bending moment at 3 is a maximum when the load is at a distance $0{\cdot}58l$ from the end. Having drawn the curve to scale, by placing a diagram of the loading on the horizontal, we get the bending moment at 3 by multiplying the loads by the corresponding ordinates; similarly for reactions. If the continuous girder consists of two spans, each 12 ft. 6 in., and if the moving load consists of a locomotive with axles 5 ft. apart, each loaded with 40,000 lbs., or 20,000 lbs. on each wheel (Fig. 285)—

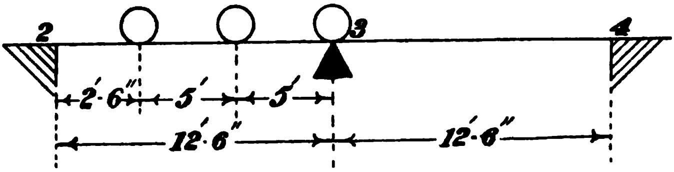


FIG. 285.

From Table L. we see that if the wheels are placed over points $k = {\cdot}2$ and $k = {\cdot}6$, the bending moment at $3 = ({\cdot}048 + {\cdot}096)$ $12{\cdot}5 \times 20{,}000 = 36{,}000$ foot pounds nearly. If the span 3-4 is also loaded, the bending moment at 3 is increased (Fig. 286).

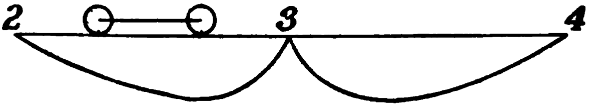


FIG. 286.

Referring to Figs. 283 and 284, we can at once calculate the reactions at each support by putting the loading over the influence line diagram. Having obtained R_2, R_3, R_4, the values of bending

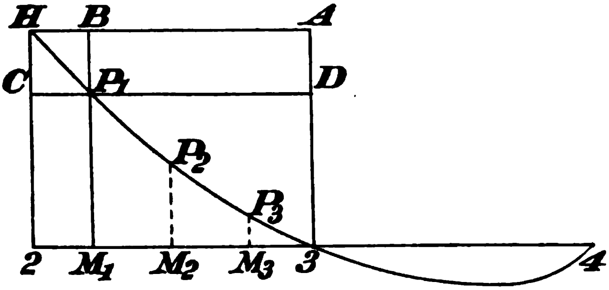


FIG. 287.

moments and shears at any point can be readily deduced. For instance, in Fig. 287 we can calculate the shear at any point F in the span 2-3 between M_3 and 3 by scaling off the ordinates P_1M_1, etc., on 2-3 and on 3-4; and taking the algebraic sum this gives R_2. The shear at F $= \Sigma\{(P_1M_1) \times \text{load at } M_1\} - \Sigma\,\{\text{load at } M_1, \text{ etc.}\}$ on left of F. Looking at Fig. 287, we get a geometrical representation of the effect of continuity in creating a bending moment at the point of support 3. Bending moment at 3 for a load at $M_1 = P_1M_1 \times (2\text{-}3) - BM_1 \times 3M_1$ where H2 represents the load to some scale—

$\therefore$ bending moment at 3 = rectangle 2CD3 − rectangle M_1BA3.
= rectangle CM_1 − rectangle P_1A.

This vanishes when the influence line HP_13 is a straight line, *i.e.*, when there is no continuity.

Three spans, the two end spans being equal in length—

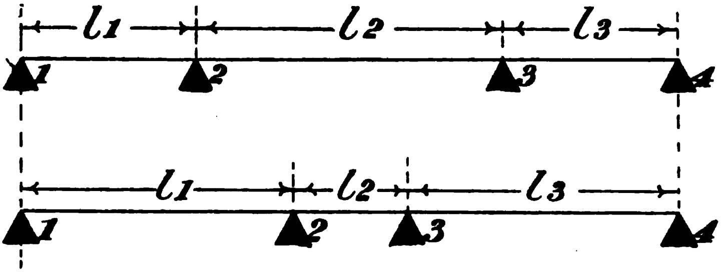


Fig. 288.

Let $l_1 = l_3 = l$ say, and $l_2 = nl$ when n may be $\gtrless$ unity. Let m represent the quantity $4(1 + n)^2 - n^2$, which will be often used. As before, $K_1 = k - k^3$ and $K_2 = (1 - k) - (1 - k)^3$.

Bending Moments.—From the general equation—

$$M_2l_2 + 2M_3(l_2 + l_3) + M_4l_3 = -\,l_2^2\,(k - k^3).$$

Load on l_1. Since $M_1 = M_4 = 0$, $2M_2\,(l_1 + l_2) + M_3l_2 = -\,l_1^2K_1$; also $M_2l_2 + 2M_3(l_2 + l_3) = 0$. $\therefore M_2 = -\dfrac{2(1 + n)}{m}K_1l$ and $M_3 = \dfrac{nK_1}{m}l$.

Load on l_2. $2M_2(l_1 + l_2) + M_3l_2 = -\,l_2^2K_2$. $M_2l_2 + 2M_3(l_2 + l_3)$ $= -\,l_2K_1$. $\therefore M_2 = \dfrac{-\,ln^2}{m}\,[2(1 + n)K_2 - nK_1]$ and

$$M_3 = \frac{-\,ln^2}{m}\,[2\,(1 + n)\,K_1 - nK_2].$$

Load on l_3. By symmetry—

$$M_2 = +\,\frac{n}{m}K_2l \text{ and } M_3 = -\,\frac{2(1 + n)}{m}K_2l.$$

Thus for the influence lines of M_2—

Load on l_1. $M_2 = \dfrac{-\,2(1 + n)}{m}K_1l$

$$\int_{k=0}^{k=1} M_2 = \frac{-\,(1 + n)}{2m}Wl.$$

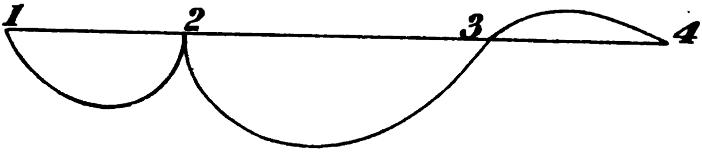
Load on l_2. $M_2 = \frac{-n^2}{m}\left\{2(1+n)K_2 - nK_1\right\}l$

$$\int_{k=0}^{k=1} M_2 = -\frac{n^2}{m} - \frac{2+n}{4}Wl.$$

Load on l_3. $M_2 = \frac{n}{m}K_2 l$

$$\int_{k=0}^{k=1} M_2 = \frac{n}{4m}Wl.$$

The influence line curve is shown in Fig. 289, also see Table LIII.



Bending Moment at 2.
FIG. 289.

On account of symmetry M_3 will be the same as M_2.
The reactions are found in the following manner:—
Reaction R_1 (Fig. 290)—

Load on l_1. $R_1 = \frac{M_2}{l} + (1-k) = (1-k) - \frac{2(1+n)}{m}K_1$

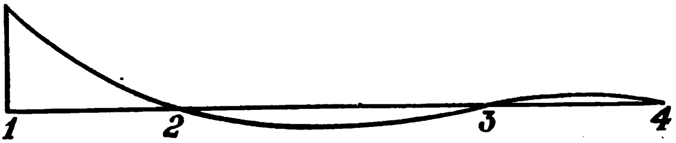
$$\int_{k=0}^{k=1} R_1 = \left\{\frac{1}{2} - \frac{1+n}{2m}\right\}W.$$

Load on l_2. $R_1 = \frac{M_2}{l} = \frac{-n^2}{m}\left\{2(1+n)K_2 - nK_1\right\}$

$$\int_{k=0}^{k=1} R_1 = -\frac{n^2}{m} \cdot \frac{2+n}{4}W.$$

Load on l_3. $R_1 = \frac{M_2}{l} = \frac{n}{m}K_2$.

$$\int_{k=0}^{k=1} R_1 = \frac{n}{4m}W.$$



Reaction at 1.
FIG. 290.

Reaction R_2—

Load on l_1. $$\mathrm{R}_2 = \left\{\frac{\mathrm{M}_3}{l} + (1 + n - k) - \mathrm{R}_1(1 + n)\right\}\frac{1}{n}$$

$$= k + (k - k^3)\left\{\frac{2(1 + n)^2 + n}{mn}\right\}.$$

$$\int_{k=0}^{k=1}\mathrm{R}_2 = \left[\frac{1}{2} + \frac{1}{4}\left\{\frac{2(1 + n)^2 + n}{mn}\right\}\right]\mathrm{W}.$$

Load on l_2. $$\mathrm{R}_2 = \left\{\frac{\mathrm{M}_3}{l} + (1 - k)n - \mathrm{R}_1(1 + n)\right\}\frac{1}{n}$$

$$\text{Where } \frac{\mathrm{M}_2}{l} = \mathrm{R}_1$$

$$= \frac{\mathrm{M}_3 - \mathrm{M}_2}{ln} - \frac{\mathrm{M}_2}{l} + (1 - k)n$$

$$= (1 - k) + \frac{n}{m}\left[\left\{2(1 + n)^2 + n\right\}\mathrm{K}_2 - (1 + n)(2 + n)\mathrm{K}_1\right]$$

$$\int_{k=0}^{k=1}\mathrm{R}_2 = \left\{\frac{1}{2} + \frac{n(2n + n^2)}{4m}\right\}\mathrm{W}.$$

Load on l_3. $$\mathrm{R}_2 = \frac{\mathrm{M}_3}{ln} - \frac{\mathrm{R}_1(1 + n)}{n} = \frac{-(1 + n)(2 + n)}{mn}\mathrm{K}_2$$

$$\int_{k=0}^{k=1}\mathrm{R}_2 = \frac{-(1 + n)(2 + n)}{4mn}\mathrm{W}.$$

On account of symmetry R_3 and R_4 will be the same as R_2 and R_1 respectively, taking k from the right end instead of from the left.

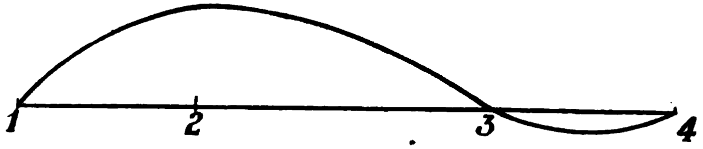
TABLE LIII.

SHOWING BENDING MOMENTS AND REACTIONS.

Figures are worked out for $n = 5/4$.

	Load on l_1.			Load on l_2.			Load on l_3.		
k	$\frac{\mathrm{M}_2}{l}$	R_1	R_2	$\frac{\mathrm{M}_2}{l}$	R_1	R_2	$\frac{\mathrm{M}_2}{l}$	R_1	R_2
	–	+	+	–	–	+	+	+	–
·05	·0120	·9380	·0743	·0296	·0296	·9960	·0062	·0619	·0290
·10	·0238	·8762	·1482	·0540	·0539	·9816	·0114	·0114	·0535
·15	·0353	·8147	·2214	·0734	·0733	·9574	·0158	·0157	·0738
·20	·0462	·7538	·2935	·0883	·0881	·9250	·0193	·0192	·0902
·25	·0564	·6936	·3641	·0989	·0989	·8850	·0219	·0219	·1027
·30	·0657	·6343	·4329	·1058	·1057	·8414	·0239	·0238	·1118
·35	·0739	·5761	·4995	·1091	·1091	·7855	·0251	·0251	·1175
·40	·0809	·5191	·5636	·1094	·1092	·7278	·0257	·0257	·1202
·45	·0864	·4636	·6248	·1068	·1066	·6662	·0257	·0256	·1201
·50	·0903	·4097	·6826	·1019	·1018	·6017	·0251	·0250	·1174
·55	·0924	·3576	·7367	·0949	·0950	·5355	·0240	·0240	·1123
·60	·0925	·3075	·7870	·0863	·0862	·4677	·0225	·0224	·1052
·65	·0940	·2596	·8328	·0763	·0762	·3996	·0205	·0205	·0961
·70	·0860	·2140	·8738	·0654	·0655	·3332	·0183	·0182	·0855
·75	·0790	·1710	·9097	·0539	·0540	·2678	·0157	·0157	·0734
·80	·0694	·1306	·9401	·0421	·0421	·2051	·0128	·0128	·0601
·85	·0568	·0932	·9649	·0305	·0305	·1464	·0098	·0098	·0459
·90	·0412	·0588	·9832	·0194	·0193	·0917	·0066	·0066	·0310
·95	·0223	·0277	·9951	·0091	·0091	·0426	·0033	·0033	·0156
1·00	·0000	·0000	1·0000	·0000	·0000	·0000	·0000	·0000	·0000

The calculation of the stresses in continuous girders is much simplified by means of influence lines as the following example will show.



Reaction at 2.

FIG. 291.

A continuous girder consisting of three equal spans of 159 ft. centres, loaded with a dead load of ·6 of a ton per foot run, and a live load of ·7 of a ton per foot run (Fig. 292).

This example has been worked out on page 226, and Case I. will now be worked by means of influence lines, in which the span 1-2 is loaded with 1·3 tons per foot run, and the spans 2-3 and 3-4 with ·6 of a ton per foot run.

It can be shown from the equation of three moments that—

$$M_2 = -\frac{4}{15}lK_1 - \frac{l}{15}(4K_2 - K_1) + \frac{l}{15}K_2$$

$$\therefore M_2 = -\frac{4l}{15}\int_{k=0}^{k=1}(k-k^3)dk - \frac{l}{15}\int_{k=0}^{k=1}(8k - 12k^2 + 4k^3)dk + \frac{l}{15}\int_{k=0}^{k=1}(k-k^3)dk$$

$+\frac{l}{15}\int_{k=0}^{k=1}(2k - 3k^2 + k^3)dk$ for unit loads, and for the actual loads—

$$\therefore M_2 = -\frac{1}{15}lW_\alpha - \frac{1}{20}lW_\beta + \frac{1}{60}lW_\gamma.$$

Where W_α = the load on span 1-2 = 159 × 1·3 tons per foot.
W_β = ,, ,, 2-3 = 159 × ·6 ,,
W_γ = ,, ,, 3-4 = 159 × ·6 ,,

$$\therefore M_2 = 159^2\left\{-\frac{1}{15}(1{\cdot}3) - \frac{1}{20}({\cdot}6) + \frac{1}{60}({\cdot}6)\right\} = 2696{\cdot}6.$$

This saves labour in deducing and solving the simultaneous equations. Instead of taking five cases, as in the example quoted, the influence lines work out the required function once for all for the structure, while without them much of the same ground is covered several times.

For the same case. Load on the span 1-2.

Reaction R_1—

$$R_1 = \frac{M_2}{l} + (1-k) = (1-k) - \frac{4}{15}(K_1)$$

$$\int_{k=0}^{k=1}R_1 = \frac{13}{30}W_\alpha.$$

Load on the span 2-3—

$$R_1 = \frac{M_2}{l} = -\frac{1}{15}\{4K_2 - K_1\}$$

$$\int_{k=0}^{k=1} R_1 = -\frac{1}{20}W_\beta.$$

Load on the span 3-4—

$$R_1 = \frac{M_2}{l} = \frac{1}{15}K_1$$

$$\int_{k=0}^{k=1} R_1 = +\frac{1}{60}W_\gamma.$$

$$\therefore R_1 = \frac{13}{30}W_\alpha - \frac{1}{20}W_\beta + \frac{1}{60}W_\gamma$$

$$= 159\left\{\frac{13}{30} \times 1{\cdot}3 - \frac{1}{20} \times {\cdot}6 + \frac{1}{60} \times {\cdot}6\right\} = +86{\cdot}39 \text{ tons.}$$

Reaction R_2.

Load on the span 1-2—

$$R_2 = \frac{M_3}{l} + (2 - k) - 2R_1 = \frac{9}{15}K_1 + k$$

$$\int_{k=0}^{k=1} R_2 = \frac{13}{20}W_\alpha.$$

Load on the span 2-3—

$$R_2 = \frac{M_3}{l} + (1 - k) - 2R_1 = (1 - k) + \frac{1}{5}(3K_2 - 2K_1)$$

$$\int_{k=0}^{k=1} R_2 = +\frac{11}{20}W_\beta.$$

Load on the span 3-4—

$$R_2 = \{(1 - k) - 3R_1 - R_3\}\frac{1}{2} = -\frac{6}{15}K_2$$

$$\int_{k=0}^{k=1} R_2 = -\frac{1}{10}W_\gamma.$$

R_3 is deduced from R_2 by symmetry, and R_4 is deduced from R_1 in the same way.

Therefore—

$$R_2 = \tfrac{13}{20}W_\alpha + \tfrac{11}{20}W_\beta - \tfrac{1}{10}W_\gamma = +\ 177{\cdot}285 \text{ tons.}$$
$$R_3 = \tfrac{13}{20}W_\gamma + \tfrac{11}{20}W_\beta - \tfrac{1}{10}W_\alpha = +\ 93{\cdot}81 \quad ,,$$
$$R_4 = \tfrac{13}{20}W_\gamma - \tfrac{1}{20}W_\beta + \tfrac{1}{60}W_\alpha = 40{\cdot}015 \quad ,,$$

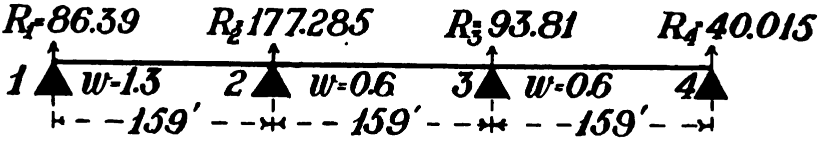
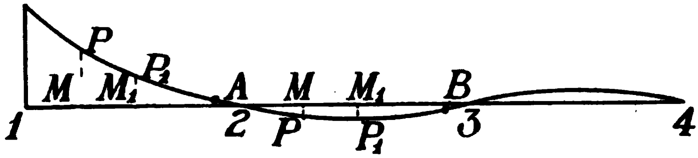


FIG. 292.

We have now the loads and reactions as shown in Fig. 292.

From this can be derived the bending moment or shear at any point and the full effect of the loading known in detail as with a simple beam.

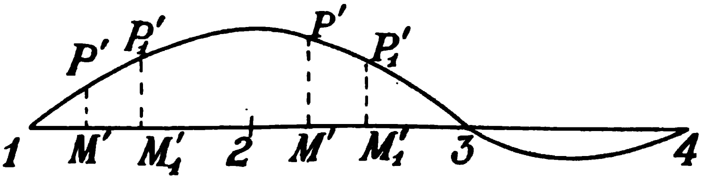
Again instead of the approximate method of determining the shearing stresses, in consequence of the change of the position of the points of contra-flexure during the passage of the rolling load, we may obtain directly the shears for any position of the rolling load by means of diagrams drawn to scale such as Figs. 293 and 294.



Reaction at 1.
FIG. 293.

For instance take a rolling load W_1, W_2 on span 1-2—
Shear at any point A on span 1-2 is—

$$\{W_1 \times PM\} + \{W_2 \times P_1M_1\} - (W_1 + W_2), \text{ Fig. 293.}$$



Reaction at 2.
FIG. 294.

Again the shear at any point B in the span 2-3 (Figs. 293 and 294) is—

$$= W_1\{PM + P'M'\} + W_2\{P_1M_1 + P_1'M'_1\} - (W_1 + W_2).$$

Take the loads to have rolled on to span 2-3, then—

$$\text{The shear at A} = -\{W_1(PM) + W_2(P_1M_1)\}, \text{ (Fig. 293).}$$

$$\begin{aligned}\text{The shear at B} = &-\{W_1(PM) + W_2(P_1M_1)\}\\ &+\{W_1(P'M') + W_2(P_1'M_1')\}\\ &-(W_1 + W_2), \text{ (Figs. 293 and 294).}\end{aligned}$$

Results could be deduced in a similar manner for concentrated and distributed loads in any position.

Mechanical Fixing of the Points of Contra-flexure and Cantilever Bridges.—We have seen that the points of contra-flexure are continually changing in position during the passage of the live load, and that in a bridge of three spans there will generally be four points of contra-flexure, *viz.*, two in the central span and one in each of the side spans. The points of contra-flexure may be fixed mechanically by uniting the girder at the proposed points by some form of hinged connection. The two points may be fixed either in the central or side spans, but not in both, or the stability of the bridge will be destroyed. The fixing of the points of contra-flexure for any span considerably simplifies the calculations for that span.

In the Kentucky bridge, United States, America, which consists of three equal spans, each 375 ft. long, there is a hinge in each of the side spans, situated at a distance of 75 ft. from the pier, thus reducing the effective length of the side spans to 300 ft. If the two hinges are located in the centre span, we have a form of the cantilever bridge in which the central span consists of two cantilevers projecting from the piers, with an independent girder resting upon their extremities and completing the span. The side spans in this case should be sufficiently long to balance the central span under all conditions of loading, otherwise the extremities must be anchored down.

The Poughkeepsie bridge over the Hudson River, United States, America, consists of cantilever and rigid spans of almost equal length arranged alternately, so that there is no necessity for anchorages, as the rigid span balances the cantilevers on each side of it.

In the more usual form of cantilever bridge the cantilevers projecting from each side are generally of equal length, so that they balance each other for the dead load; the independent girder rests on the extremities of the cantilevers as before, while the extremities of the side cantilevers must be anchored down to balance the central portion under all conditions of loading.

The ratio of the length of the independent girder to that of the cantilevers varies in different bridges. Examples of this type occur in the bridges over the St. John's, Niagara, and the Frazer rivers, and also the Red Rock cantilever bridge, United States, America.

The celebrated bridge over the Forth consists of two cantilever spans, each 1700 feet long; the cantilevers project 680 ft. on each side of the piers, with a maximum depth of 343 ft.; the independent central girders are each 340 ft. span. The Forth bridge has been fully illustrated in the various engineering journals, and its detailed description will not be attempted here. It is in every respect the greatest constructional achievement in the world.

The cantilever bridge is one of the most economical types for long spans. The calculations may be made by first considering the independent girder, the reactions of which upon the extremities of the cantilevers must be combined with the panel loads of the cantilever.

It will be generally most convenient to consider each panel, proceeding from the extremity of the cantilevers step by step to the pier. Owing to the varying depth, the panel loads due to the weight of the panels are not equal.

Cantilever bridges possess the advantage of being easily erected by building outwards from the piers, and are specially applicable for long spans over deep gorges or rivers, where ordinary scaffolding would be too expensive or subject to great risks. Cantilever bridges should not generally be less than 500 ft. span.

Fig. 295 represents a proposed cantilever bridge designed by Fives-Lille Co., France, and Fig. 296 a similar bridge designed by the Maschinenbau-Aktien-Gesellschaft, Nürnberg. Both bridges were submitted for crossing Sydney Harbour,[1] and represent the best modern practice in the design of such bridges. In Fig. 295 the anchor spans *ab* and *ef* are each 591 ft., the cantilever arms *bc* and *de* are each 328 ft., and the suspended span *cd* is 590 ft. long. The proportions adopted in Fig. 296 are different, and the top member forms an unbroken curve. In cantilever bridges the suspended span *cd* is treated precisely in the same manner as an ordinary detached span, but the connections at *c* and *d* are designed so that free expansion and contraction may take place, denoting the lengths of the portions *ab*, *bc*, and *cd*, by l_1, l_2, and l_3 respectively, and assuming a uniform dead load of w per lineal foot over the whole bridge, which would not be true in Figs. 295 and 296, since the dead load must increase from the centre of *cd* to the piers *b* and *e*. Let x denote the distance from *c* in the portion l_2, then the shear and bending moment at this section of the cantilever arm are—

$$S = -\left(\frac{wl_3}{2} + wx\right), \quad M = -\left(\frac{wl_3x}{2} + \frac{wx^2}{2}\right).$$

For a single load on l_3 at a distance kl_3 from C, we have—

$$S = -Wk, \quad M = -Wkx.$$

For a single load on l_2 at a distance of x' from the section in l_2, we have—

$$S = -W, \quad M = -Wx'.$$

For the anchor span l_1, let R_2 denote the reactions at *a* and *b* respectively, and kl_1 the distance of a single load W in the position l_1 from the point *a*, then—

$$R_1 = W(1-k), \quad R_2 = Wk.$$

For a single load W on the cantilever arm l_2, at a distance of kl_2 from the point *b*, the reactions are—

$$R_1 = -W\frac{kl_2}{l_1}, \quad R_2 = W + W\frac{kl_2}{l_1}.$$

The negative reaction R_1 may be greater than the positive reaction due to loads on the cantilever arm and anchor span respectively, necessitating anchoring down the end *a*. In Fig. 296 the anchor span is 65 ft.

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[1] *Report of the Sydney Harbour Bridge Advisory Board.*

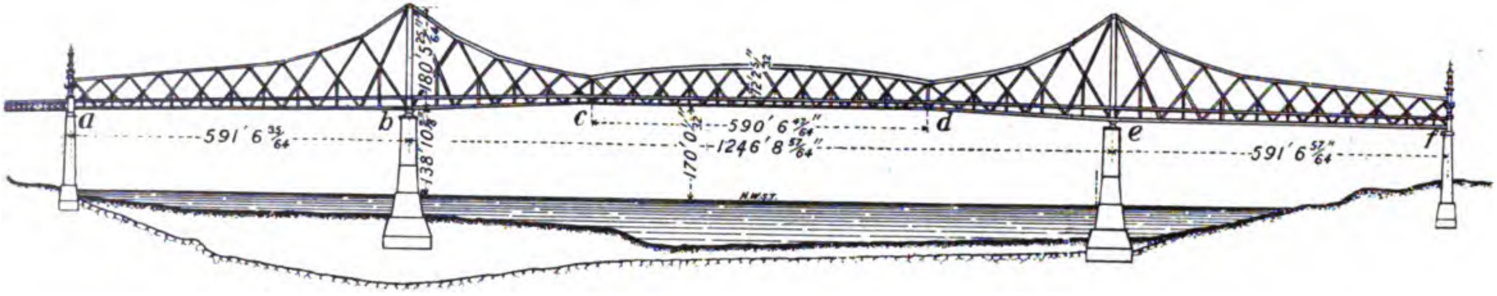

FIG. 295.

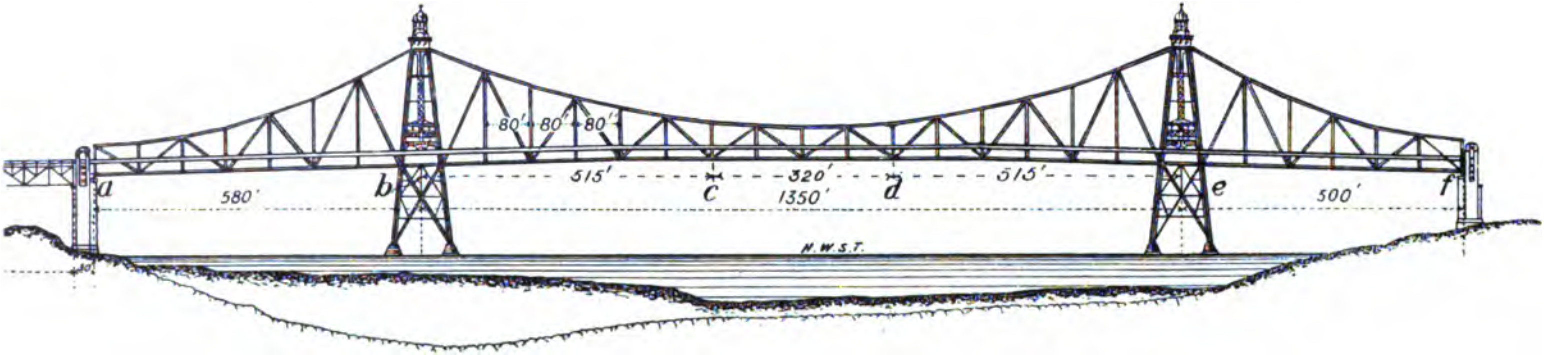

FIG. 296.

longer than the cantilever arm, but the dead load reaction at *a* is negative; also the reaction due to a live load extending from *b* to *d* must be negative. The live load on this bridge for two railway and two tramway tracks, also 55 ft. of roadway and footpaths, may be taken at about 5 tons per lineal foot, therefore the negative reaction for this load over the position *bd* only is—

$$R_1 = -\left(\frac{5\times320}{2}+\frac{5\times515}{2}\right)\frac{515}{580} = -1853 \text{ tons.}$$

The negative reaction due to the dead load must be added to that produced by the live load, and the total negative reaction must be provided for by suitable anchorages. In Fig. 295 it may be shown that the negative reaction at *a* for similar loading is about 1273 tons. Messrs. Merriman[1] and Jacoby have investigated the most economical proportions of l_1, l_2, l_3, where local conditions do not influence the position of the piers, and if L denotes the total length of the bridge *af*, they have expressed the results as follows:—

$$l_1 = 0{\cdot}212\text{L},\ l_2 = 0{\cdot}091\text{L},\ l_3 = 0{\cdot}394\text{L}.$$

These results can only be regarded as approximations, but they show that for a uniform load over the whole bridge the negative reaction at *a* is zero.

If $\text{L} = 2500$ ft., then $l_1 = 530$, $l_2 = 227{\cdot}5$, and $l_3 = 985$ ft. If we compare these values with those of Figs. 295 and 296 we see that the cantilever arm is shorter, and the suspended arm longer than shown in Figs. 295 and 296. In the Forth bridge, according to this rule, the values would be 648·72, 278·46 and 1205·64 ft. respectively for the anchor span, cantilever arm, and suspended span. Cantilever bridges should be carefully considered in regard to the stresses developed in the various members during erection, more especially when the bottom member is curved as in the Quebec bridge. If a crane is used to erect the bridge, the stresses due to its load, and the weight of the structure acting as a cantilever, should not be able to reverse the stresses in any of the diagonal tension members. This reversal of stress will not occur if the centre of moments, for the members on the left of the centre, is never on the left of the centre of the suspended span. If the centre of moments for any diagonal tension member on the left, for instance, should occur on the left of the centre of the suspended span, it must be temporarily counterbraced to prevent reversal of stress when the crane is in the most unfavourable position.[2]

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[1] *Higher Structures*, Part iv.

[2] *Proc. Inst. C.E.*, vol. clxxiii., Prof. Claxton Fidler.

CHAPTER XII.

STRENGTH OF COLUMNS.

Short Columns.—Fig. 297 represents a rectangular prism subjected to compressive stress between the flat plates of a testing-machine. If the compressive force is applied along the axis of the prism, and if P denote the total load, A the area of the prism, and p_1 the intensity of stress, then—

$$\frac{\mathrm{P}}{\mathrm{A}} = p_1.$$

The stress p_1 is uniformly distributed over the area of the section.

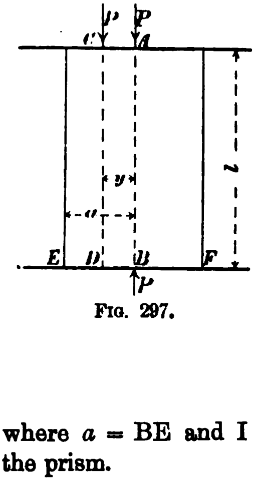


FIG. 297.

If the load is applied at some point other than the centre of gravity of the section, such as C, Fig. 297, we may conceive two opposite forces applied along the axis AB equal to p; then we have a uniform compressive stress over the area of the column—

$$p_1 = \frac{\mathrm{P}}{\mathrm{A}},$$

also a couple, the moment of which is—

$$\mathrm{M} = \mathrm{P}y.$$

The maximum compressive stress p_2 due to the couple Py will occur at E, and will be—

$$p_2 = \frac{\mathrm{P}ya}{\mathrm{I}},$$

where a = BE and I is the moment of inertia of the cross-section of the prism.

If r denote the radius of gyration, then—

$$p_2 = \frac{\mathrm{P}ya}{\mathrm{A}r^2}.$$

Hence the maximum intensity of compressive stress at E is—

$$p = \frac{\mathrm{P}}{\mathrm{A}}\left(1 + \frac{ya}{r^2}\right).$$

Long Columns.—In a long column the ratio of length to the least radius of gyration may be sufficient to cause the column to fail by lateral flexure rather than by direct crushing, in which case we have a compound stress somewhat similar to that existing in the short columns with eccentric loading. The tendency to lateral flexure in a long column may be increased by the eccentricity of the loading caused by the direction of the load not coinciding with the true axis of the column. The strength of long columns has been investigated mathematically by Euler and Rankine, Prof. R. Smith, Prof. Claxton Fidler and others, and experimentally by Hodgkinson, Christie and at the Watertown Arsenal, United States, America.

Euler's Formulæ for Columns.—Euler's theory of the strength of columns may be stated as follows :—

Fig. 298 represents a column fixed in direction at one end only, which bends as shown in the figure.

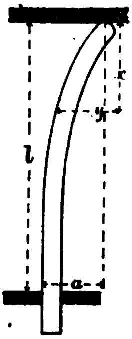


FIG. 298.

If the column fails by direct crushing, then—

$$P_1 = fA.$$

If by flexure, then it can be proved that—

$$P_2 = \frac{\pi^2}{4l^2}EI\,^{1} \quad . \quad . \quad . \quad . \quad (1)$$

---

[1] Let ρ = radius of curvature, M = bending moment at xy, then—

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{Py}{EI}; \ \frac{1}{\rho} = -\frac{d^2y}{dx^2}$$

$$\therefore -\frac{d^2y}{dx^2} = \frac{P}{EI}y$$

multiply each side by $\int \frac{dy}{dx}dx$—

$$-\int \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} \cdot dx = \frac{P}{EI}\int y \cdot \frac{dy}{dx} \cdot dx$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = -\frac{P}{EI}y^2 + C$$

and since for $y = a, \frac{dy}{dx} = 0$; $\therefore C = \frac{P}{EI}a^2$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{P}{EI}(a^2 - y^2)$$

$$\frac{dy}{dx} = \sqrt{\frac{P}{EI}}\sqrt{a^2 - y^2}$$

$$\therefore \int \frac{dy}{\sqrt{a^2 - y^2}} = \int \sqrt{\frac{P}{EI}} \cdot dx$$

$$\therefore \sin^{-1}\frac{y}{a} = \sqrt{\frac{P}{EI}}x + C$$

and since when $x = 0, y = 0$; $\therefore C = 0$

$$\text{and } \sin^{-1}\left(\frac{y}{a}\right) = \sqrt{\frac{P}{EI}} \cdot x.$$

When $y = a$, we have $x = l$.

$$\text{Hence } \frac{\pi}{2} = \sqrt{\frac{P}{EI}} \cdot l$$

$$\text{and } P = \frac{\pi^2}{4l^2} \cdot EI.$$

Then, according to Euler, the actual breaking strength will be the smaller of the two values given for P_1 and P_2 in the foregoing equations.

If the column is rounded or hinged at each end (Fig. 299), then—

$$P_2 = \frac{\pi^2}{l^2} EI \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(here length = $2l$ in (1)).

If the column is fixed at each end (Figs. 300 and 300A), either by building in or by means of flat ends—

$$P_2 = \frac{4\pi^2}{l^2} EI \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(here length = $4l$ in (1)).

If the column is fixed in position at each end and one end is fixed in direction (Fig. 301)—

$$P_2 = \frac{\pi^2}{4l^2} EI$$

(here length = $3l$ in (1)).

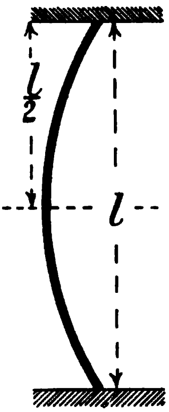


FIG. 299.

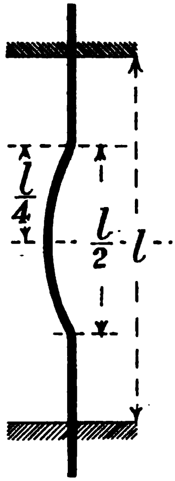


FIG. 300.

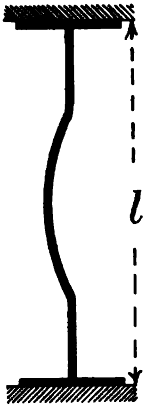


FIG. 300A.

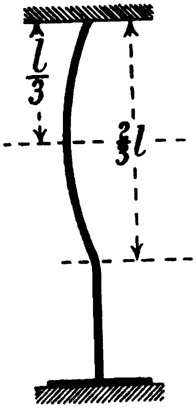


FIG. 301.

These formulæ are strictly true only when applied to an ideal column consisting of uniformly elastic material in which the axis of the straight column coincides with the line of pressure of the load. Prof. Smith and Mr. Claxton Fidler[1] have, however, pointed out that these conditions cannot be realised in practice, as there is nearly always some eccentricity due to imperfections of workmanship, want of uniformity in the modulus of elasticity throughout the column, journal friction, occurring for instance in connecting and eccentric rods. The strength of columns is therefore more a matter of probability than of exact theory.

---

[1] Prof. R. H. Smith in *Engineering*, 14th October, 1887; Mr. Claxton Fidler, *Min. Proc. Inst. C.E.*, vol. lxxxvi.

In the case of a column hinged at each end $P = \frac{\pi^2}{l^2}EI$, and, dividing by the area, we have—

$$\rho = \frac{P}{\text{area}} = \pi^2 E \frac{r^2}{l^2}$$

where r denotes the radius of gyration. If we plot the curve represented by this equation, using $\frac{l}{r}$ for abscissæ, and ρ for ordinates, we obtain BHC (Fig. 302), which represents the higher limit of strength.

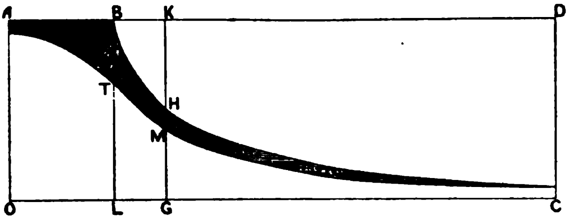


FIG. 302.

The straight line ABK represents the constant value of f, the ultimate crushing strength. GH denotes the stress due to the direct load; HK, the stress due to the ultimate deflection or bending moment.

In the case of a girder-shaped strut, such as would be used as a compression member in a bridge, Mr. Claxton Fidler points out that if the ends are rounded, the eccentricity due to the variation in the modulus may be treated in the following manner :—

Let ϵ_1 = the specific compression in the inner flange $= \frac{1}{E_1}$.

ϵ_2 = „ „ „ „ outer „ $= \frac{1}{E_2}$.

ϵ = average compression $= \frac{\epsilon_1 + \epsilon_2}{2}$.

δ = the central deflection due to load P.

y = the distance from the neutral axis to the extreme fibre.

θ = the eccentricity.

r = the radius of gyration = half the depth in a girder-shaped strut.

$p = \frac{P}{A}$ = direct load in pounds per sq. in.

$$\text{Then } \theta = r\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2},$$

and it can be proved that—

$$\delta = \frac{\pi r}{2} \cdot \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \cdot \frac{p}{\rho - p} = \frac{\theta\pi}{2} \cdot \frac{p}{\rho - p}.$$

$$\text{Since } \mathrm{M} = \mathrm{P}\delta = \frac{f_1 \mathrm{I}}{y} = \frac{f_1 \mathrm{A} r^2}{y}$$

$$\pm f_1 = \frac{\mathrm{P}\delta y}{\mathrm{A}r^2} = \frac{p\delta y}{r^2}.$$

$$\therefore \pm f_1 = \frac{\pi}{2} \cdot \frac{y}{r} \cdot \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \cdot \frac{p^2}{\rho - p}.$$

$$\text{If } \phi = \frac{y}{r} \cdot \frac{\pi}{2} \cdot \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}$$

$$\pm f_1 = \frac{\phi p^2}{\rho - p}$$

$$\text{and since } f = p + f_1 = p\left(1 + \frac{\phi p}{\rho - p}\right)$$

$$\text{hence } p = \frac{f + \rho - \sqrt{(f + \rho)^2 - 4f\rho(1 - \phi)}}{2(1 - \phi)}$$

This equation, when plotted as before, using $\frac{l}{r}$ for abscissæ, gives a wavy line, ATMC, which Mr. Fidler calls the curve of the lower limit of the strength of the column, and he states that "between the shaded area included between the two curves, the results of individual experiments," plotted in a similar manner, "may be expected to range themselves at haphazard. The upper limit will be the line of the ideal column ABC, and it remains to determine for each material the position of the lower limit ATC, which must evidently be regarded as the greatest reliable strength of the column."

The diagram, Fig. 302, shows that the strength of a long column depends more upon the modulus of elasticity than upon the crushing resistance of the material, thus: GM denotes the stress due to the direct load, and MK the stress due to bending moment. The diagram also shows that a very long or very short column will not differ much from the ideal column, but that for intermediate values of $\frac{l}{r}$ the effect of eccentricity, or inequality in the modulus of elasticity, becomes most marked; the maximum effect occurs at B, where $\rho = f$, and

$$p = f\frac{1 - \sqrt{\phi}}{1 - \phi}.$$

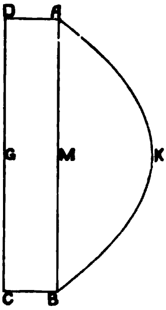


FIG. 303.

In Fig. 303, GM denotes the stress at the centre of the column due to direct load, and MK that due to bending moment; hence the graphical representation of the stress in the column is as shown in Fig. 303, and if the column is braced, the stresses in the lattice bars may be deduced from the flange stresses, as shown by the curve AMBK.

Mr. Fidler has calculated the strength of columns for various values of the ratio $\frac{l}{r}$, by means of the formulæ referred to, and he states that the results agree closely with those obtained from experiments made by Mr. Christie and others.

These results are tabulated on page 254.

Hodgkinson's Formulæ for Columns.[1]—The principal practical use of Euler's formulæ was to furnish a general form of expression for the breaking load to the experiments of Eaton Hodgkinson.

Applying Euler's formula to solid cylindrical columns with flat ends, and substituting for $I=\frac{\pi r^4}{4}=\frac{\pi d^4}{64}$, we have—

$$P_1=\left(\frac{2\pi}{l}\right)^2 EI=\tfrac{1}{16}E\pi^3\frac{d^4}{l^2}.$$

But Hodgkinson's experiments gave—

$$P_1=(\text{constant})\times\frac{d^{3{\cdot}55}}{l^2}\text{ for wrought-iron,}$$

$$P_1=(\text{constant})\times\frac{d^{3{\cdot}55}}{l^{1{\cdot}7}}\text{ for cast-iron.}$$

For a square column, in which the length of side is denoted by b, Euler's formula would give—

$$P_1=\frac{\pi^2}{3}E\frac{b^4}{l^2},$$

and Hodgkinson's experiments on timber gave—

$$P_1=(\text{constant})\times\frac{b^4}{l^2}.$$

Inserting the constants determined by Hodgkinson, his formulæ become, when the length exceeds 30 times the diameter—

$P_1=\frac{44d^{3{\cdot}55}}{l^{1{\cdot}7}}$ for cast-iron columns with flat ends, where l is in feet and d in inches, and P_1 the breaking load in tons.

$P_1=\frac{134d^{3{\cdot}55}}{l^2}$ for wrought-iron columns, ditto, ditto.

$P_1=\frac{10{\cdot}95b^4}{l^2}$ for dry oak, ditto, ditto.

$P_1=\frac{7{\cdot}81b^4}{l^2}$ for dry fir, ditto, ditto.

For hollow columns of cast-iron we have, where D and d denote the external and internal diameters respectively in inches—

$$P_1=44\frac{D^{3{\cdot}55}-d^{3{\cdot}55}}{l^{1{\cdot}7}}.$$

---

[1] See straight line equation for cast-iron columns, page 274.

TABLE LIV.

STRENGTH OF COLUMNS WITH ROUND ENDS IN POUNDS PER SQUARE INCH OF SECTIONAL AREA, ACCORDING TO MR. CLAXTON FIDLER.

Ratio $\frac{l}{r}$	Cast-iron.	Wrought-iron.	Mild steel.	Hard steel.
20	72,300	35,200	46,700	67,200
40	50,800	32,600	42,700	58,600
60	30,000	28,400	36,000	45,500
80	17,600	23,200	28,300	33,000
100	11,700	18,200	21,500	23,700
120	8,300	14,100	16,400	17,500
140	6,300	11,100	12,700	13,300
160	4,900	8,800	10,100	10,400
180	3,900	7,200	8,160	8,360
200	3,200	5,900	6,710	6,850
220	2,680	4,970	5,620	5,710
240	2,270	4,210	4,750	4,820
260	1,950	3,640	4,080	4,130
280	1,690	3,140	3,550	3,570
300	1,480	2,750	3,100	3,130
320	1,300	2,430	2,730	2,740
340	1,160	2,160	2,430	2,440
360	1,040	1,940	2,190	2,190
380	940	1,730	1,960	1,960
400	850	1,570	1,760	1,760

TABLE LV.

STRENGTH OF COLUMNS WITH FIXED ENDS IN POUNDS PER SQUARE INCH OF SECTIONAL AREA, ACCORDING TO MR. CLAXTON FIDLER.

Ratio $\frac{l}{r}$	Cast-iron.	Wrought-iron.	Mild steel.	Hard steel.
20	77,600	35,800	47,400	68,700
40	67,800	34,900	45,700	65,800
60	54,700	33,400	43,300	60,500
80	42,000	31,100	39,900	53,600
100	30,000	28,400	36,000	45,500
120	21,200	25,300	31,000	37,400
140	16,000	22,200	26,500	30,500
160	12,600	19,200	22,500	25,000
180	10,200	16,500	19,100	20,900
200	8,300	14,100	16,400	17,500
220	6,900	12,100	13,900	14,900
240	5,700	10,500	12,000	12,600
260	5,000	9,300	10,400	11,000
280	4,400	8,200	9,100	9,500
300	3,900	7,200	8,200	8,400
320	3,400	6,300	7,200	7,300
340	3,000	5,600	6,300	6,500
360	2,700	5,100	5,500	5,700
380	2,470	4,600	5,100	5,200
400	2,270	4,210	4,750	4,800

Hodgkinson found that the strength of columns with flat ends, one end flat and the other rounded, both ends rounded, are as 3 : 2 : 1, approximately.

Rankine's Formula.—This formula is based on the assumption that the load acts along the axis of the column, and that there is no eccentricity. It may be investigated in the following manner :—[1]

Let θ = the eccentricity.
P = the load.
δ = the central deflection.
R = the radius of curvature.
A = the area of the cross-section.
r = the radius of gyration.
f_1 = the intensity of stress due to bending within the elastic limit of the material.
f = the intensity of stress at rupture.
y = the distance from the neutral axis to the extreme fibre.
l = the length of the column.

It can be proved that, in a column fixed at one end and free at the other—

$$R = \frac{l^2}{2\delta} \text{ approximately,}$$

but—

$$\frac{1}{R} = \frac{M}{EI} = \frac{P(\delta+\theta)}{EI}$$

$$\therefore \frac{2\delta}{l^2} = \frac{P(\delta+\theta)}{EI}$$

$$2\delta EI = P(\delta+\theta)l^2$$

$$\therefore \delta = \frac{P\theta l^2}{2EI - Pl^2}, \quad \text{or} \quad \delta + \theta = \frac{2EI\theta}{2EI - Pl^2},$$

but since $M = P(\delta + \theta) = \dfrac{f_1 I}{y}$

$$f_1 = \frac{P(\delta+\theta)y}{I} = \frac{2PE\theta y}{2EI - Pl^2}.$$

But $f = f_1 + \dfrac{P}{A}$

$$\therefore f_1 = f - \frac{P}{A} = \frac{2PE\theta y}{2EI - Pl^2}$$

$$\text{or } fA = P\left(1 + \frac{2E\theta y}{2EI - Pl^2}\right)$$

or $P(2EI - Pl^2 + 2E\theta yA) = fA(2EI - Pl^2)$.

Since Pl^2 is small compared with $2EI + 2AE\theta y$, we may neglect it.

$$\therefore \frac{P}{A} = \frac{2fEI}{2E(I + A\theta y) + fAl^2} = \frac{f}{1 + \frac{\theta y}{r^2} + \frac{fl^2}{2Er^2}}$$

---

[1] Dubois.

If $\theta = 0$—

$$\frac{\mathrm{P}}{\mathrm{A}} = \frac{f}{1 + \frac{f}{2\mathrm{E}} \cdot \frac{l^2}{r^2}} = \frac{f}{1 + c\frac{l^2}{r^2}}$$

c is a constant depending on the material and the conditions of fixing, or otherwise, at the ends. This is Rankine's formula, and it is clear that it is only true when there is no eccentricity, and when the stress produced is within the elastic limit of the material. In order not to exceed the elastic limit, the intensity of buckling stress, $\frac{\mathrm{P}}{\mathrm{A}}$, should be divided by a suitable factor of safety. If the factor of safety is expressed thus, $\frac{1}{4 + \frac{l}{20d}}$, which is convenient for variable loads, where d is the least dimension of the cross-section, then, if we denote the working stress by W, we have—

$$\mathrm{W} = \frac{1}{4 + \frac{l}{20d}}\left(\frac{f}{1 + \frac{cl^2}{r^2}}\right)$$

The usual factor of safety for dead loads is four.

Gordon's Formula.—This formula is of the same form as Rankine's, but, instead of using the least radius of gyration, r, the least dimension of the cross-section is substituted, and the value of c modified accordingly. This will be true for circular and rectangular sections, since r is a function of d, but for all other sections r is not a simple function of d, hence Rankine's formula is more general than Gordon's.

In Rankine's formula f is taken as 36,000 for wrought-iron, and $c = \frac{1}{36000}$ for flat ends, $= \frac{1}{9000}$ for round ends, $= \frac{1}{20000}$ for one end flat and the other end round.

Professor Bauschinger recommends the following formulæ for columns in fire-proof buildings, as the result of a long series of experiments in which the columns were heated and sprinkled with water :—

$$\frac{\mathrm{P}}{\mathrm{A}} = \frac{14223}{1 + 0{\cdot}0001\frac{l^2}{r^2}} \text{ for wrought-iron,}$$

$$\frac{\mathrm{P}}{\mathrm{A}} = \frac{19912}{1 + 0{\cdot}00025\frac{l^2}{r^2}} \text{ for cast-iron,}$$

where $l =$ the length, and $r =$ the least radius of gyration.

The tests of full-sized columns (pp. 260 to 266) were made at the Watertown Arsenal in 1881.

The columns were placed in the testing-machine on accurate bearings formed by inserting thin pieces of rolled brass between the ends of the column and the face of the holder when there was not a perfect fit.

Counterpoise weights were attached to the long columns at two points of their length, to counteract the tendency to bend from the weight of the column. Figs. 304 to 309 show the construction of some of these columns.

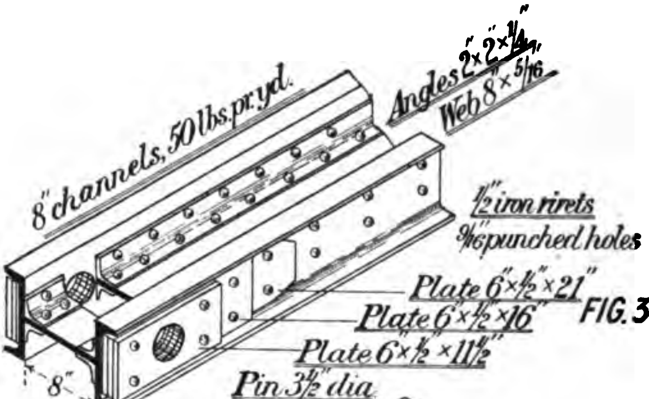


FIG. 304

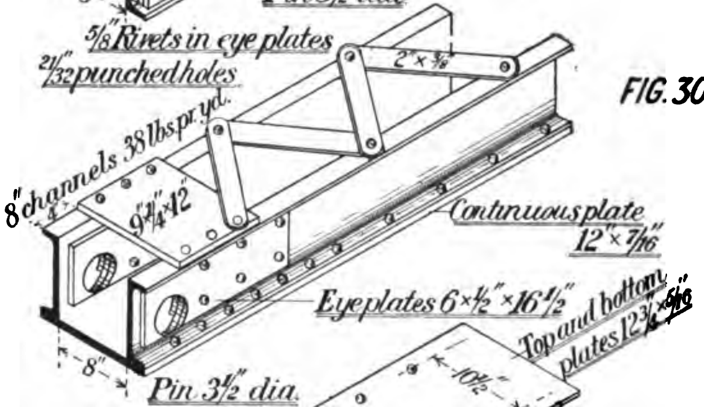

5/8" Rivets in eye plates
21/32" punched holes
2"×3/8"
8" channels 38 lbs. pr. yd.
8"
9 1/4"×12"
Continuous plate 12"×7/16"
Eye plates 6"×1/2"×16 1/2"
Top and bottom plates 12 3/4"×5/16"
8"
Pin 3 1/2" dia.

FIG. 305

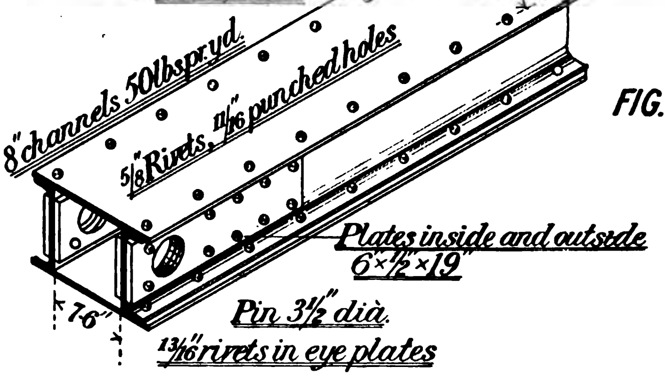

10 1/2"
8" channels 50 lbs. pr. yd.
5/8 Rivets, 11/16" punched holes
Plates inside and outside 6"×1/2"×19"
7·6"
Pin 3 1/2" dia.
13/16" rivets in eye plates

FIG. 306

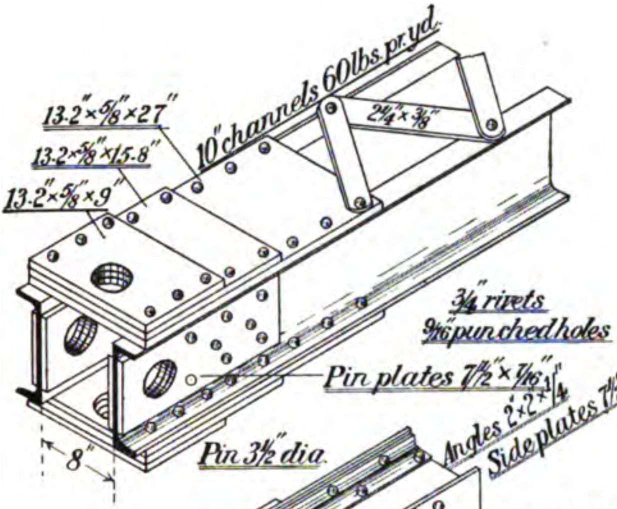


FIG. 307

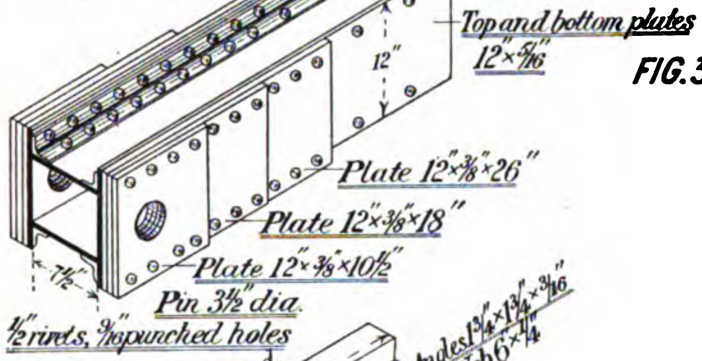


FIG. 308

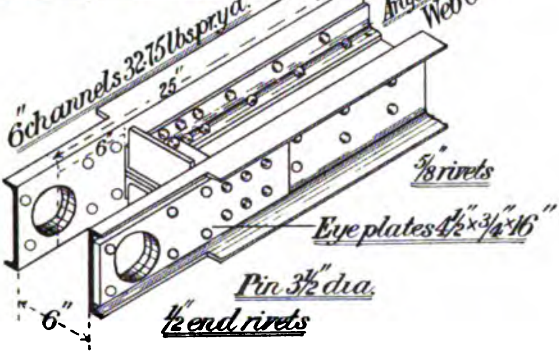


FIG. 309

TABLE LVI.

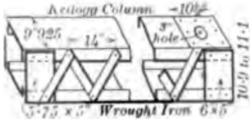
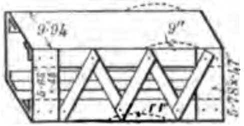
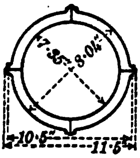
No. of test.	Style of column.	Length in ft. and in.	Sectional area in sq. in.	Ultimate strength.		Elastic limit in pounds per sq. in.	Remarks.
				Total pounds.	Pounds per sq. in.		
1	 Kellogg Column —10½ 9·925 14" 3" hole 10·9 to 11·1 5·75 × 5" Wrought Iron 6×5 FIG. 310.	29′ 9·3″	19·244	480,000	25,980	—	Tested with cap at end A, and pin of hard cast steel 3 inches in diameter at end B. Plate on pin end cracked at middle rivet, also channel beam under this crack.
2		27′ 9″	19·4473	550,000	28,280	22,620	Cap and pin not used; both ends take square bearings on compression plate.
3	„	19′ 9″	19·486	590,000	30,740	24,120	Tested in a similar manner to No. 2. Failed suddenly, springing upwards, breaking every brace on each side, excepting in the middle of the column.
4	„	15′ 9″	19·3012	460,000	25,130	—	Cap not used. Pin end tested with pin used in No. 1 column. Cracked upper channel beam through middle rivet, then lower channel on side of hole near pin end.
5	„	11′ 9″	19·5614	498,000	25,460	25,050	Tested in a similar manner to No. 4. Failed at lower channel beam, a crack extending from pin 5 in. long through centre rivet-hole.
6	„	8′ 0″	19·1026	574,500	30,080	24,600	Tested as in Nos. 4 and 5. Maximum deflection 1·15 in. End braces gave out at rivet-holes.
7	 FIG. 311.	4′ 0″	19·3616	720,000	37,180	24,270	The column deflected upwards as shown in dotted lines.

Table LVII.

No. of test.	Style of column.	Length in ft. and in.	Sectional area in sq. in.	Ratio $\frac{l}{r}$	Ultimate strength.		Elastic limit in pounds per sq. in.	Deflection in inches.	
					Total pounds.	Pounds per sq. in.		Horizontal.	Vertical.
1	 Fig. 312.	28′ 0″	12·062	112	424,200	35,168	—	4·552	3·269
2		28′ 0″	12·180	112	416,000	34,150	—	0·05	2·47
3		25′ 0″	12·233	100	431,500	35,270	27,960	—	—
4		25′ 0″	12·100	100	424,000	35,040	—	1·715	0·067
5		22′ 0″	12·371	88	440,000	35,570	—	0·077	0·006
6		22′ 0″	12·311	88	423,000	34,360	—	0·010	0·391
7		19′ 0″	12·023	76	425,200	35,365	—	0·092	0·024
8		19′ 0″	12·087	76	446,000	36,900	29,290	0·016	0·032
9		16′ 0″	12·000	64	439,000	36,580	—	2·09	0·002
10		16′ 0″	12·000	64	439,000	36,580	—	3·22	0·012
11		13′ 0″	12·185	52	449,000	36,857	28,890	1·435	0·284
12		13′ 0″	12·069	52	449,000	37,200	—	2·196	0·336
13		10′ 0″	12·248	40	446,823	36,480	26,940	0·049	0·030
14	„	10′ 0″	12·339	40	449,000	36,397	28,360	0·036	0·016
15	„	7′ 0″	12·265	28	468,000	38,157	29,350	0·535	—
16	„	7′ 0″	11·962	28	517,000	43,300	29,590	2·110	0·846
17	„	4′ 0″	12·080	16	598,000	49,500	—	—	—
18	„	4′ 0″	12·119	16	621,000	51,240	—	—	—
19	„	0′ 8″	11·903	2·7	680,000	57,130	—	—	—
20	„	0′ 8″	11·903	2·7	682,000	57,300	—	—	—

Compression of wrought-iron columns, lattice, box and solid web, made at Watertown Arsenal in 1883 and 1884.

This series of tests comprises seventy-four columns made by the Detroit Bridge and Iron Company.

Tabulation of experiments with 3½-inch pins in the centre of gravity of cross-section, except those marked N, in which the loading was eccentric. Sufficient bearing area for the pins and flat ends was provided by means of reinforcing plates.

TABLE LVIII.

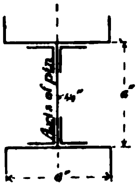
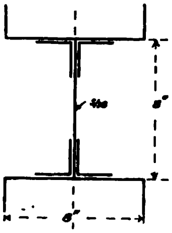
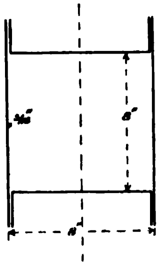
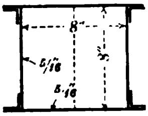
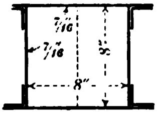
No. of test.	Style of column.	Length centre to centre of pins in in.	Sectional area in sq. in.	Ratio $\frac{l}{r}$	Ultimate strength.		Manner of failure.
					Total pounds.	Pounds per sq. in.	
752	 FIG. 313.	120·20	9·831	64	297,100	30,220	Deflected perpendicular to axis of pins.
757		120·07	10·199	65	320,000	31,380	Sheared rivets in eye-plates.
755		180·00	9·977	96	251,000	25,160	Deflected perpendicular to axis of pins.
756		180·00	9·977	96	210,000	21,050	Ditto.
753		240·00	9·732	128	188,600	19,380	Ditto.
754		240·00	9·762	128	158,300	16,220	Ditto.
751	 FIG. 314.	240·00	16·077	98	425,000	26,430	Deflected perpendicular to axis of pins.
1642		240·00	16·281	98	367,000	22,540	Ditto.
1646		320·00	16·179	131	318,800	19,700	Ditto.
1647		320·10	16·141	131	283,600	17,570	Ditto.
1653	 FIG. 315.	320·00	17·898	90	474,000	26,480	Deflected perpendicular to axis of pins.
1654		320·00	19·417	90	491,000	25,290	Ditto.
1645	 FIG. 316.	319·95	16·168	96	453,000	28,020	Deflected parallel to axis of pins.
1650		320·00	16·267	96	454,000	27,910	Deflected perpendicular to axis of pins.
1651	 FIG. 317.	320·00	20·954	98	540,000	25,770	Deflected in diagonal directions.
1652		320·10	20·613	98	535,000	25,950	Sheared rivets in eye-plates.

TABLE LVIII.—*continued.*

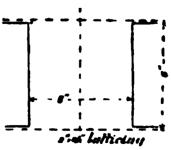
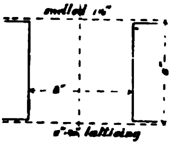
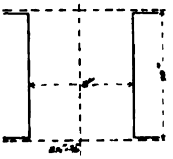
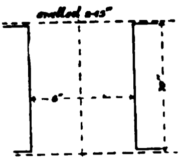
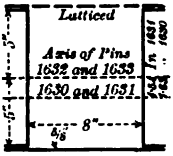
No. of test.	Style of column.	Length centre to centre of pins in in.	Sectional area in sq. in.	Ratio $\frac{l}{r}$	Ultimate strength.		Manner of failure.
					Total pounds.	Pounds per sq. in.	
746	 FIG. 318.	159·20	7·628	53	258,700	33,910	Deflected perpendicular to axis of pins.
747		159·27	8·056	78	294,700	36,580	Ditto.
748		239·60	7·621	79	260,000	34,120	Ditto.
749		239·60	7·621	79	254,600	33,410	Deflected diagonally.
1648		319·90	7·705	107	243,600	31,610	Deflected parallel to axis of pins.
1649		319·85	7·673	107	229,200	29,870	Deflected diagonally.
740	 FIG. 319.	159·90	7·645	53	262,500	34,340	Deflected perpendicular to axis of pins.
741		159·90	7·624	53	255,650	33,530	Ditto.
739		239·70	7·517	79	251,000	33,390	Deflected parallel to axis of pins.
750		239·70	7·531	79	259,000	34,390	Deflected perpendicular to axis of pins.
1643		319·80	7·691	107	237,200	30,840	Deflected parallel to axis of pins.
1644		319·92	7·702	107	237,000	30,770	Deflected horizontally.
1640	 FIG. 320.	199·84	11·944	52	403,000	33,740	Deflected perpendicular to axis of pins.
1641		200·00	12·302	52	426,500	34,670	Deflected diagonally.
1634		300·00	12·148	78	408,000	33,630	Deflected perpendicular to axis of pins.
1635		300·00	12·175	78	395,000	32,440	Ditto.
1638	 FIG. 321.	199·25	12·366	52	385,000	31,130	Deflected perpendicular to axis of pins.
1639		199·50	12·659	52	405,000	31,990	Ditto.
1636		300·20	11·920	70	391,400	32,830	Deflected in diagonal direction.
1637		300·15	11·932	70	390,700	32,740	Ditto.
1630	 Latticed Axis of Pins 1632 and 1633 1630 and 1631 8" FIG. 322.	300·00	17·622	75	461,500	26,190	Deflected perpendicular to axis of pins.
1631		300·00	17·231	76	485,000	28,150	Ditto.
1632		300·00	17·570	78	306,000	17,420	Ditto.
1633		300·00	17·721	78	307,000	17,270	Ditto.

TABLE LIX.

WROUGHT-IRON COLUMNS WITH FLAT ENDS.

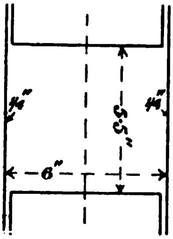
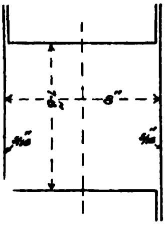
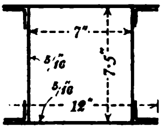
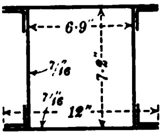
No. of test.	Style of column.	Total length in in.	Sectional area in sq. in.	Ratio $\frac{l}{r}$	Ultimate strength.		Manner of failure.
					Total pounds.	Pounds per sq. in.	
377	 FIG. 323.	127·900	12·08	47	383,200	31,722	Buckling of ½-inch plate between riveting.
378		127·900	11·11	47	372,900	33,564	Buckling plates.
379	 FIG. 324.	167·80	17·01	47	594,500	34,950	Buckling plates between the riveting.
380		167·80	17·80	47	633,600	35,595	Triple flexure.
346	 FIG. 325.	167·90	15·74	51	517,000	32,846	Buckling plates.
347		167·90	15·84	51	555,200	35,050	Ditto.
342		247·62	15·68	75	517,500	33,003	Deflected upwards.
344		247·80	15 56	75	536,900	34,505	Buckling plates.
348	 FIG. 326.	167·75	21·02	51	708,000	33,682	Buckling plates.
349		167·75	21·46	51	709,500	33,061	Triple flexure.
341		247·60	21·20	75	700,000	33,019	Deflected upwards
343		247 62	21·49	76	729,450	33,943	Deflected downwards.

TABLE LIX.—*continued.*

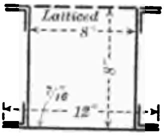
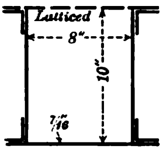
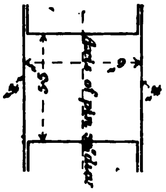
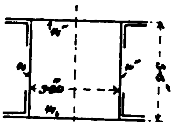
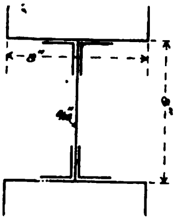
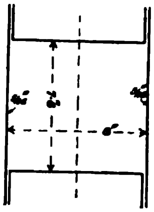
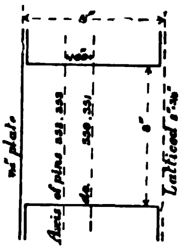
No. of test.	Style of column.	Total length in in.	Sectional area in sq. in.	Ratio $\frac{l}{r}$	Ultimate strength.		Manner of failure.
					Total pounds.	Pounds per sq. in.	
339	 FIG. 327.	247·94	12·64	79	412,900	32,666	Deflected upwards, latticing on the concave side.
340		247·94	12·74	79	431,400	33,862	Ditto.
337	 FIG. 328.	307·75	16·90	76	582,400	34,279	Deflected downwards and sideways, continuous plate on the concave side.
338		307·87	17·40	77	580,000	33,333	Deflected diagonally, latticing on the concave side.

TABLE LX.

WROUGHT-IRON COLUMNS WITH 3½-INCH PINS.

No. of test.	Style of column.	Length centre to centre of pins in in.	Sectional area in sq. in.	Ratio $\frac{l}{r}$	Ultimate strength. Total pounds.	Ultimate strength. Pounds per sq. in.	Manner of failure.
368	 FIG. 329.	180·10	11·42	65	379,200	33,205	Horizontal deflection perpendicular to plane of pins.
367		180·00	11·42	65	369,200	32,329	Ditto.
356		240·00	11·42	87	342,000	29,947	Ditto.
357		240·00	11·31	87	330,100	29,186	Ditto.
371	 FIG. 330.	119·90	9·14	49	286,100	31,302	Buckling the plates between the rivets.
372		120·00	10·07	49	319,200	31,698	Ditto.
370		180·00	9·21	74	291,500	31,650	Horizontal deflection and buckling between the rivets.
369		180·00	9·44	74	290,000	30,720	Ditto.
354		240·00	9·24	98	267,500	28,950	Triple flexure.
355		240·00	9·36	98	279,700	29,879	Horizontal deflection.
360	 FIG. 331.	160·12	15·34	65	475,000	30,965	Deflected upwards in the plane of the 3½-in. pins.
361		160·00	15·40	65	485,000	31,494	Horizontal deflection perpendicular to the plane of the pins.
358	 FIG. 332.	240·00	17·77	67	570,000	32,077	Horizontal deflection perpendicular to the plane of the pins.
359		240·00	17·22	67	555,400	32,253	Ditto.
350	 FIG. 333.	240·25	12·48	79	202,700	16,242	Horizontal deflection, concave on lattice side.
351		240·00	10·84	79	208,200	19,207	Ditto.
352		240·25	12·65	77	350,000	27,668	Ditto.
353		240·25	12·76	77	390,400	30,596	Horizontal deflection perpendicular to the plane of the pins, convex on lattice side.

Moncrieff[1] has investigated the strength of columns in a very complete manner. After the manner of Prof. Fidler he investigated formulæ and plotted curves representing the upper and lower limits of strength of columns; the upper limit on the curves is below the curve representing the ideal column in Fidler's diagrams. Moncrieff has also plotted the results of all the experiments on columns published up to 1900, and which range themselves between his upper and lower limiting curves.

If P is the load applied at the ends of a column of length l with an eccentricity ϵ, from the axis of a column at its ends, producing a deflection at the centre of the length of the column Δ, Fig. 334. Then since the curve of flexure of a column of uniform cross-section is a curve of sines, whether loaded eccentrically or not, under actual conditions this curve differs very slightly from a parabola, so that the value of Δ may be shown to be—

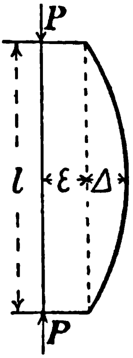


FIG. 334.

$$\Delta = \frac{Pl^2\epsilon}{8EI - \frac{5}{6}Pl^2}.$$

This equation is equal to infinity if the value of the eccentricity is small, and the denominator is zero—

$$8EI - \tfrac{5}{6}Pl^2 = 0 \therefore P = \frac{9{\cdot}6EI}{l^2}.$$

In Euler's equation—

$$P = \frac{\pi^2}{l^2}EI = \frac{9{\cdot}87EI}{l^2}.$$

So that by assuming the curve to be a parabola, we make an error of 2·73 per cent. on the safe side.

Let f_b = the intensity of stress due to bending moment.
f_d = the intensity of stress due to direct compression.
c = the distance from the neutral axis to the extreme fibre.

Then—

$$M = P(\epsilon + \Delta) = \frac{f_b I}{c} = \frac{Ar^2 f_b}{c}$$

$$\therefore f_b = \pm\frac{P(\epsilon + \Delta)c}{Ar^2}.$$

Where I = moment of inertia, r = radius of gyration, A = area of cross-section,

$$f_d = \frac{P}{A}.$$

---

[1] The practical column under central or eccentric loads, *Trans. Am. Soc. C.E.*, vol. xlv., June, 1901, by J. M. Moncrieff, M.Inst.C.E., M.Am.S.C.E.

The total stress is therefore—

$$F = f_d \pm f_b = \frac{P}{A} \pm \frac{P(\epsilon + \Delta)c}{Ar^2}.$$

Substitute for Δ, then—

$$F = \frac{P}{A} \pm \left\{\frac{P\left(\epsilon + \frac{Pl^2\epsilon}{8EI - \frac{5}{6}Pl^2}\right)c}{Ar^2}\right\}$$

$$= \frac{P}{A}\left\{1 \pm \frac{c}{r^2}\left(\frac{Pl^2\epsilon}{8EI - \frac{5}{6}Pl^2} + \epsilon\right)\right\}$$

$$= f_d\left\{1 \pm \frac{c\epsilon}{r^2}\left(\frac{f_d l^2}{8Er^2 - \frac{5}{6}f_d l^2} + 1\right)\right\}$$

If we put $R = \frac{l}{r}$, then it can be shown that—

$$F = f_d\left\{1 \pm \frac{c\epsilon}{r^2}\left(\frac{f_d R^2}{8E - \frac{5}{6}f_d R^2} + 1\right)\right\}$$

$$= f_d\left\{1 \pm \frac{c\epsilon}{r^2}\left(\frac{48E + f_d R^2}{48E - 5f_d R^2}\right)\right\}$$

Using the + sign in the brackets to determine the maximum fibre stress F_c—

$$F_c = f_d\left\{1 + \frac{c\epsilon}{r^2}\left(\frac{48E + f_d R^2}{48E - 5f_d R^2}\right)\right\}$$

from which we have by simple algebraic transformation, the value of R corresponding to any fixed value of the maximum compressive stress F_c—

$$R = \sqrt{\frac{48E}{5F_c + f_d\left(\frac{c^2}{r^2} - 5\right)}\left[\frac{F_c}{f_d} - 1 - \frac{c\epsilon}{r^2}\right]}$$

In a similar manner using the − sign we obtain the minimum fibre stress (not necessarily tensile) F_t.

$$F_t = f_d\left\{1 - \frac{c\epsilon}{r^2}\left(\frac{48E + f_d R^2}{48E - 5f_d R^2}\right)\right\}$$

which is easily transformed to—

$$R = \sqrt{\frac{48E}{5F_t - f_d\left(\frac{c\epsilon}{r^2} + 5\right)}\left[\frac{F_t}{f_d} - 1 + \frac{c\epsilon}{r^2}\right]}$$

These two equations for R give the general relations of the column proportion to the column strength, and it will be observed that there are three factors to which it is necessary to assign values, these are:

1. The value of E.
2. The value of F_c or F_t = the maximum or the minimum working fibre stress allowable.
3. The value of $\frac{c\epsilon}{r^2}$.

Let E = 30,000,000 pounds per sq. in. for steel.

Let F_c = 24,000 pounds per sq. in., approximately as shown by the higher tests of short columns.

F_t is derived from direct tests of tensile strength.

Moncrieff found that the upper limit of strength is given fairly by the formulæ when—

$$\frac{c\epsilon}{r^2} = 0{\cdot}15,$$

and the lower limit when—

$$\frac{c\epsilon}{r^2} = 0{\cdot}6.$$

These values are applicable to cast-iron, wrought-iron, mild and hard steel, also to timber columns.

The factor of safety K against stability may be taken at $\frac{1}{3}$, then—

$$f_d = \frac{P}{A} = \tfrac{1}{3}\,\frac{9{\cdot}6Er^2}{l^2} = 3{\cdot}2\,\frac{Er^2}{l^2}.$$

The table on the following page gives some examples of the use of Moncrieff's formulæ in the design of columns under central loading.

Columns under Intentionally Eccentric Loads.—Moncrieff's formulæ are directly applicable to columns with eccentric loading by simply adding the measured eccentricity in the expression, $\frac{c\epsilon}{r^2}$, to the value of $\frac{c\epsilon}{r^2} = 0{\cdot}6$, as determined for centrally loaded columns; thus the formulæ for mild steel columns become for—

$$\frac{c\epsilon}{r^2} - 0{\cdot}6 = 2, \text{ then } \frac{l}{r} = 10\sqrt{\frac{480}{12 - 3f_d}\left(\frac{2{\cdot}4}{f_d} - 3\right)}$$

$$\text{for } \frac{c\epsilon}{r^2} - 0{\cdot}6 = 8, \text{ then } \frac{l}{r} = 10\sqrt{\frac{480}{12 + 3f_d}\left(\frac{2{\cdot}4}{f_d} - 9\right)}$$

The Strength of Columns as Affected by the End Conditions.—A column with fixed or free ends cannot be completely realised in practice. The nearest approach to a free-ended column is found in the ordinary pin connections used largely in American truss bridges, but the pin nearly fits the hole, and the frictional resistance developed, depending upon the closeness of fit, smoothness of bearing surfaces, tends to prevent lateral deflection and adds to the resistance of the column in proportion to the ratio of the diameter of the pin to the radius of gyration of the column.

TABLE LXI.

MONCRIEFF'S WORKING FORMULAE FOR COLUMNS UNDER "CENTRAL LOADING".

(1)	(2)	(3)	(4)	(5)
Material of column.	$F_c=$ Maximum compressive fibre stress. Pounds per sq. in. dead load.	KE= $\dfrac{\text{Mod. Elas.}}{3}$ Pounds per sq. in.	Value of $\dfrac{c\epsilon}{r^2}$	Working formulæ for round-ended or pivot-ended columns. $\dfrac{l}{r} = \sqrt{\dfrac{48KE}{5F_c + f_d\left(\dfrac{c\epsilon}{r^2} - 5\right)}\left[\dfrac{F_c}{f_d} - 1 - \dfrac{c\epsilon}{r^2}\right]}$
Common cast-iron	12,000 = 1·2 units.	$\dfrac{14,000,000}{3}$ = 466·6 units.	0·6	$\dfrac{l}{r} = 10\sqrt{\dfrac{224}{6 - 4{\cdot}4f_d}\left(\dfrac{1{\cdot}2}{f_d} - 1{\cdot}6\right)}$
Wrought-iron . . . . Tensile strength, 45,000 to 50,000 lbs. per sq. in.	18,000 = 1·8 units.	$\dfrac{28,000,000}{3}$ = 933·3 units.	0·6	$\dfrac{l}{r} = 10\sqrt{\dfrac{448}{9 - 4{\cdot}4f_d}\left(\dfrac{1{\cdot}8}{f_d} - 1{\cdot}6\right)}$
Mild steel . . . . Tensile strength, 60,000 to 70,000 lbs. per sq. in.	24,000 = 2·4 units.	$\dfrac{30,000,000}{3}$ = 1,000 units.	0·6	$\dfrac{l}{r} = 10\sqrt{\dfrac{480}{12 - 4{\cdot}4f_d}\left(\dfrac{2{\cdot}4}{f_d} - 1{\cdot}6\right)}$
Hard steel . . . . Tensile strength, about 100,000 lbs. per sq. in.	36,000 = 3·6 units.	$\dfrac{30,000,000}{3}$ = 1,000 units.	0·6	$\dfrac{l}{r} = 10\sqrt{\dfrac{480}{18 - 4{\cdot}4f_d}\left(\dfrac{3{\cdot}6}{f_d} - 1{\cdot}6\right)}$
	Timber (Unit for F_c, f_d and E = 1,000 lbs.).			
Yellow pine or pitch pine . .	2,000 = 2 units.	$\dfrac{2,200,000}{3}$ = 733·3 units.	0·6	$\dfrac{l}{r} = 10\sqrt{\dfrac{352}{10 - 4{\cdot}4f_d}\left(\dfrac{2}{f_d} - 1{\cdot}6\right)}$
White pine	1,300 = 1·3 units.	$\dfrac{1,400,000}{3}$ = 466·6 units.	0·6	$\dfrac{l}{r} = 10\sqrt{\dfrac{224}{6{\cdot}5 - 4{\cdot}4f_d}\left(\dfrac{1{\cdot}3}{f_d} - 1{\cdot}6\right)}$
French oak or Dantzic oak . .	2,000 = 2 units.	$\dfrac{1,200,000}{3}$ = 400 units.	0·6	$\dfrac{l}{r} = 10\sqrt{\dfrac{192}{10 - 4{\cdot}4f_d}\left(\dfrac{2}{f_d} - 1{\cdot}6\right)}$

In Column (5) a unit of 10,000 lbs. is used for F_c, f_d and E for metals, and 1,000 lbs. for timber.

The nearest approach to a fixed end occurs in columns with ends squarely abutting upon a rigid or fixed base, which can only apply to short columns. In long columns, the distribution of pressure over the base may be such that tension is developed on the edge of the base near the convex side of the deflected column, in which case the ends are no longer fixed. In fixed ends the bending moment at the ends is about equal to that at the centre, whereas with free ends the bending moment at the ends is zero.

A column with fixed ends carrying a given load may be twice the length of a free-ended column of the same cross-section and having the same stresses, only in the case of the ideal column. In bridge work the ends are approximately fixed in proportion to the relative stiffness of the column to the members attached to it.

The fixity of riveted connections in consequence of secondary stresses may be more apparent than real, and pin connections are sometimes stiffer. It is clear that so long as tensile stresses are not set up in a flat-ended column, it will behave as one with fixed ends.

Moncrieff obtains the value of l/r when tensile stress is just commencing by making $F_t = 0$, then—

$$\frac{l}{r} = 2\sqrt{\frac{48E}{0 - f_d\left(\frac{c\epsilon}{r^2} + 5\right)}\left[0 - 1 + \frac{c\epsilon}{r^2}\right]}$$

$$= 2\sqrt{\frac{48E\left(1 - \frac{c\epsilon}{r^2}\right)}{f_d\left(\frac{c\epsilon}{r^2} + 5\right)}}.$$

Fig. 335 represents the results of testing 390 columns with flat and fixed ends, of which only 25 were fixed. The tests plotted by Moncrieff comprise the following:—

41 tests by Hodgkinson, rectangular, flat ends.
61 „ „ Christie and Davies, and at Watertown, channels and beams, flat ends.
50 „ „ Christie and Davies, angle bars, flat ends.
49 „ „ Hodgkinson and Christie, tubes, flat ends.
39 „ „ Christie and Davies, tie bars, flat ends.
79 „ „ various experimenters, large-size columns, flat ends.
13 „ „ Bauschinger, various sections, flat ends.
20 „ „ C. A. Marshall, solid round, and rectangular, flat ends.
25 „ „ Christie, angle bars, fixed ends.

The curves were plotted from the following data:—

$E = 28{,}000{,}000$ pounds per sq. in.
$F_c = 48{,}000$ pounds per sq. in.
$\frac{c\epsilon}{r^2} = 0\cdot6$, for lower limit curve.
$\frac{c\epsilon}{r^2} = 0\cdot15$, for upper limit curve.
$f_d =$ average compressive load in pounds, per sq. in.

The formulæ for the upper and lower limits used are—

$$\text{Fixed ends, } \frac{l}{r} = 2\sqrt{\frac{48E}{5F_c + f_d\left(\frac{c\epsilon}{r^2} - 5\right)}\left[\frac{F_c}{f_d} - 1 - \frac{c\epsilon}{r^2}\right]}$$

for failure by compression;

$$\text{Flat ends, } \frac{l}{r} = 2\sqrt{\frac{48E\left(1 - \frac{c\epsilon}{r^2}\right)}{f_d\left(\frac{c\epsilon}{r^2} + 5\right)}}.$$

for incipient tension.

The weakness of flat-ended columns in the higher ratios of l/r is most clearly evident in the diagram, Fig. 335. The high strength shown by a number of columns of ratio $\frac{l}{r} = 30$, and less, indicates that the true column action did not come into play definitely at these small ratios. In short columns the useful ultimate strength should be measured by the elastic limit in compression. Moncrieff has plotted the upper and lower limiting curves of columns with hinged ends, and the results of all published tests of cast-iron, wrought-iron,

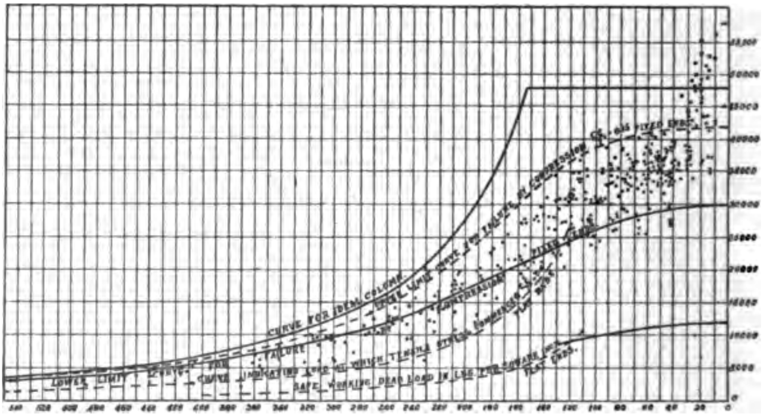


FIG. 335.

mild and hard steels, as well as of timber columns, all fall between these curves, constructed for fixed, flat, or hinged ends, as the case may be. In regard to the experiments recorded in Tables LVIII. to LX. the four columns purposely loaded out of the centre, Nos. 1632, 1633, 350 and 351 fell below the lower limiting curve. In some cases Moncrieff has shown that the radius of gyration was greater about an axis perpendicular to the pins, than parallel to the pins, and that the greater radius should be used, since these columns were practically flat ended in regard to failure in the direction of least radius of gyration. A column may be weaker as a flat-ended column deflecting parallel to the pins, than as a pin-ended column deflecting at right angles to the pins.

Remarks on Column Formulæ.—The full-size tests of columns recorded in Tables LVI. to LX. furnish the most reliable data for the determination of the strength of columns, and Mr. Theodore Cooper has proposed formulæ of the Rankine type for these tests which represent the experimental results better than Rankine's, thus:—

For lattice columns with pin ends—

$$\frac{P}{A}=\frac{34000}{1+\dfrac{\left(\dfrac{l}{r}-60\right)^2}{12000}}$$

For box and solid web columns with flat ends—

$$\frac{P}{A}=\frac{33000}{1+\dfrac{\left(\dfrac{l}{r}-80\right)^2}{10000}}$$

Mr. C. G. Schneider of the American Bridge Company has adopted the Rankine type of formula for the determination of the safe working strength of columns in soft and medium steel; thus if we denote the safe working stress in pounds per sq. in. by σ, we have—

$$\sigma=\frac{15000}{1+\dfrac{1}{13500}\dfrac{l^2}{r^2}}\text{ for soft steel.}$$

$$\sigma=\frac{17000}{1+\dfrac{1}{11000}\dfrac{l^2}{r^2}}\text{ for medium steel.}$$

In the American Bridge Company's specification for highway bridges, the results are tabulated for different values of $\frac{l}{r}$ from 10 to 140, but no compression member is allowed in which the ratio $\frac{l}{r}$ exceeds 120. The same formulæ are used for railway bridges, but the live load stresses are increased for impact and train momentum, so as to represent equivalent dead loads. The sectional area required is found by dividing the equivalent dead load by σ.

Straight Line Formulæ.—It is assumed in straight line formulæ that the results of the experiments within working limits of $\frac{l}{r}$ may be expressed with sufficient accuracy for the practical design of columns by means of a linear equation. In cast-iron columns the results of tests of larger columns than Hodgkinson's by Messrs. Ewing and Henning may be expressed thus:—

$$\sigma = 30,500 - 160\frac{l}{d}, \text{ flat ends.}$$

$\frac{l}{d}$ must not be less than 10 or greater than 35.

The working stress for such columns with a factor of safety of 5 is—

$$\sigma = 6100 - 32\frac{l}{d}.$$

This formula represents the strength of full-size cast-iron columns better than those derived by Hodgkinson within the limits assigned.

Such formulæ have the advantage of simplicity and convenience in application to the design of columns, and they have been proposed by Prof. Burr and Mr. Cooper for the compression members of bridges. They cannot be compared with the general formulæ of Moncrieff, and they must not be used outside the limits of the ratio $\frac{l}{r}$, but may be safely used within these limits, thus:—

For phœnix columns, $\frac{l}{r} = 30$ to 140—

$$\sigma = 39,640 - 46\frac{l}{r}.$$

For flat and fixed ends, angles and tees, $\frac{l}{r} = 40$ to 200—

$$\sigma = 44,000 - 140\frac{l}{r}.$$

For hinged end iron angles and tees, $\frac{l}{r} = 40$ to 200—

$$\sigma = 46,000 - 175\frac{l}{r}.$$

For flat-end iron channels and girder sections between the limits $\frac{l}{r} = 20$ to 240—

$$\sigma = 40,000 - 110\frac{l}{r}.$$

The foregoing formulæ represent the results of experiments fairly well, and they require to be divided by a suitable factor of safety to determine the safe working intensity of stress.

It appears safe and consistent to increase the intensity of stress in steel columns by about 20 per cent. that of wrought-iron columns of similar construction.

Cooper[1] gives the following formulæ for medium steel columns in railway bridges:—

For chord segments—

$$\sigma = 10{,}000 - 45\frac{l}{r} \text{ live load stresses.}$$

$$\sigma = 20{,}000 - 90\frac{l}{r} \text{ dead load stresses.}$$

All posts of through bridges—

$$\sigma = 8500 - 45\frac{l}{r} \text{ live load stresses.}$$

$$\sigma = 17{,}000 - 90\frac{l}{r} \text{ dead load stresses.}$$

All posts of deck bridges and trestles—

$$\sigma = 9000 - 40\frac{l}{r} \text{ live load stresses.}$$

$$\sigma = 18{,}000 - 80\frac{l}{r} \text{ dead load stresses.}$$

Lateral struts and rigid bracing—

$$\sigma = 13{,}000 - 60\frac{l}{r} \text{ wind stresses.}$$

But he limits the length of the compression members to the least radius of gyration at 100 for main members and 120 for laterals. Members subjected to alternating stresses of tension and compression shall be proportioned to resist each kind of stress. Both the stresses shall be considered as increased by an amount equal to 80 per cent. of the smaller of the two stresses.

Timber Columns.—Straight line formulæ may be used in the design of timber columns:—

For yellow pine in railway structures the safe working stress is—

$$\sigma = 750 - 9\frac{l}{d}; \ \frac{l}{d} = 20 \text{ to } 60.$$

For white pine—

$$\sigma = 500 - 6\frac{l}{d}; \ \frac{l}{d} = 30 \text{ to } 60.$$

---

[1] *General Specifications for Railway and Highway Bridges*, by Theodore Cooper.

For short columns of yellow pine where $\frac{l}{d}$ is less than 20 and in similar columns of white pine where $\frac{l}{d}$ is less than 30, we may use—

Yellow pine—

$\sigma = 550$ for railway bridges.
$\sigma = 1100$ for temporary structures.

White pine—

$\sigma = 300$ for railway bridges.
$\sigma = 600$ for temporary structures.

METHOD OF DESIGNING A PIN-ENDED COLUMN.

The method of designing a pin-ended column will be illustrated in the following example of a vertical compression member 50 ft. long between the centres of the pins, of medium steel, subjected to loads equivalent to a total dead load of 280,000 pounds (Figs. 336 to 338).

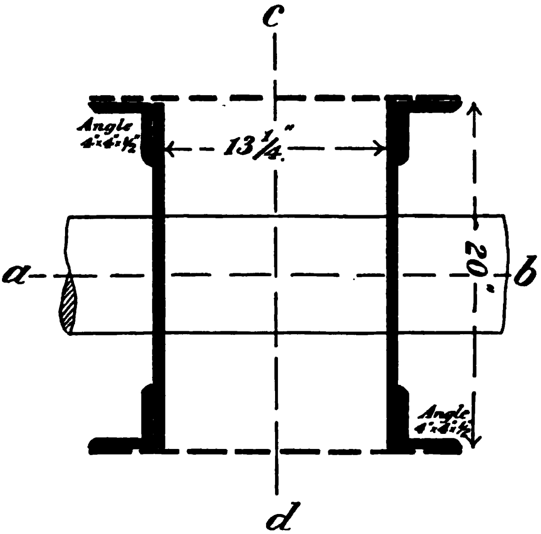


FIG. 336.

We will use Cooper's straight-line formula, which is satisfactory up to $\frac{l}{r} = 100$—

$$\sigma = 17{,}000 - 90\frac{l}{r}.$$

The least width should be about one-thirtieth of the length, or—

$$\frac{600}{30} = 20 \text{ in.}$$

The column must be built up with rolled sections arranged so that the moment of inertia about the axis at right angles to that of the pin

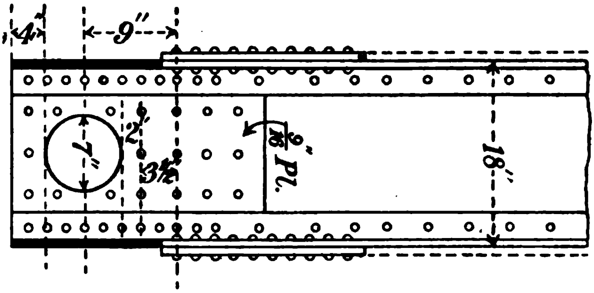


Fig. 337.

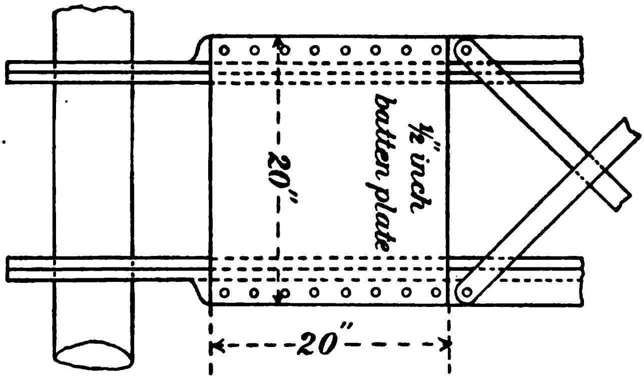


Fig. 338.

should be slightly greater than about the axis parallel with the pin. For the first trial assume that the radius of gyration is—

$$r = 0{\cdot}35 \times 20 = 7.$$

Assume also for the first trial that the width between the plates is 12 in., then—

$$\sigma = 17{,}000 - 90\frac{600}{7} = 9285 \text{ pounds per sq. in.}$$

The area required is therefore—

$$\frac{280000}{9285} = \text{about } 30 \text{ sq. in.}$$

The following plates and angles provide a gross area of 30 sq. in.—

Four angles each $4 \times 4 \times \frac{1}{2} = 15$ sq. in.
Two plates „ $20 \times \frac{3}{8} = 15$ „

We must now calculate the moment of inertia about the axes *ab* and *cd*. About *ab*—

$$\text{Two plates } 20 \times \tfrac{3}{8} = \qquad 2 \times \tfrac{3}{8} \times \frac{\overline{20}^3}{12} \qquad = \quad 500{\cdot}00$$

$$\text{Four angles } 4 \times 4 \times \tfrac{1}{2} = \tfrac{1}{12}\{8(\overline{20}^3 - \overline{19}^3) + (\overline{19}^3 - \overline{12}^3)\} = 1188{\cdot}25$$

$$\text{Moment of inertia} = 1688{\cdot}25$$

The moment of inertia about *od* is found in a similar manner to be 1454—which is less than about the axis *ab*—so that we must increase the distance between the plates. If we make this distance 13·25 in. the moment of inertia about *cd* will be 1710, or sufficiently greater than about *ab*. The least radius of gyration is therefore—

$$r = \sqrt{\frac{1688}{30}} = 7{\cdot}5$$

$$\text{and } \sigma = 17{,}000 - 90\frac{600}{7{\cdot}5} = 9{,}800 \text{ pounds per sq. in.}$$

The area required is—

$$\frac{280000}{9800} = 28{\cdot}8 \text{ sq. in.}$$

Hence the arrangement of plates and angles shown in Figs. 336 to 338 may be safely adopted. The diameter of the pin must be determined in the manner explained in Chap. XIII., but it will be taken as 8 in. The safe intensity of bearing and shearing stress will be taken as 20,000 and 10,000 pounds per sq. in. respectively. The total thickness of metal in the two jaws must be—

$$2t = \frac{280000}{8 \times 20000} = 1{\cdot}75 \text{ in.}$$

It will be necessary to provide reinforcing plates $\frac{1}{2}$-in. thick riveted to the side plates to give the necessary bearing area on the pin, and it is very desirable to make these jaws strong and stiff, so that $\frac{3}{8}$-in. plates will be used. The bearing value of the $\frac{3}{8}$-in. side plates on the pin is—

$$8 \times \tfrac{3}{8} \times 20{,}000 = 60{,}000 \text{ pounds.}$$

The stress on the rivets above the pin on each side is—

$$140{,}000 - 60{,}000 = 80{,}000 \text{ pounds.}$$

The shearing value of a $\frac{7}{8}$ rivet is less than its bearing value on a $\frac{3}{8}$ plate, so that this value will govern the number of rivets in the reinforcing plate above the pin—

$$\frac{80000}{6000} = 13{\cdot}3, \text{ say 14 rivets.}$$

Lacing Bars.—The size of the lacing bars uniting the two sides of the column is more a matter of judgment than calculation, but the following method may be used where it is desired to find the stress on these bars (Fig. 334). The bending moment is—

$$P(\epsilon + \Delta).$$

Moncrieff has shown that the eccentricity corresponding with his lower limit curve is given by the equation—

$$\frac{c\epsilon}{r^2} = 0{\cdot}6. \quad \therefore \epsilon = \frac{\overline{7{\cdot}5}^2 \times 0{\cdot}6}{6{\cdot}5} = 5{\cdot}2 \text{ in.}$$

Assume that $\epsilon = 5$ in., then—

$$\Delta = \frac{Pl^2\epsilon}{8EI - \frac{5}{6}Pl^2} = \frac{280 \times 36 \times 5}{8 \times 2{\cdot}8 \times 1688 - 280 \times 30}.$$

The quantities are divided by 10,000,000—

$$\therefore \Delta = 1{\cdot}8 \text{ in. nearly,}$$
$$\text{and } \epsilon + \Delta = 6{\cdot}8 \text{ ,, \quad ,,}$$
$$M = 280 \times 6{\cdot}8 = 1904 \text{ units of 1000 inch pounds.}$$
$$\therefore \frac{Wl}{8} = 1904 \quad \therefore S = \frac{W}{2} = \frac{1904 \times 4}{600} = 12{\cdot}7 \text{ units.}$$

The stress on the lacing = 12·7. Sec $\theta = 12{\cdot}7 \times 1{\cdot}414$. The net area required is—

$$\frac{12{\cdot}7 \times 1{\cdot}414}{17} = \frac{17{\cdot}96}{17} = 1{\cdot}06 \text{ sq. in.}$$

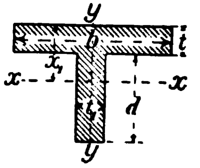
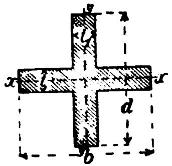
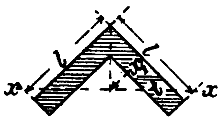
Lattice bars $2\frac{1}{2} \times \frac{3}{8}$ on each side provide a net area of—

$$2(2{\cdot}5 - {\cdot}8)\,\tfrac{3}{8} = 1{\cdot}27 \text{ sq. in.}$$

Batten plates are arranged on each side at the ends 22 in. × 22 in. × ½ in., riveted with ¾ rivets as shown.

The following tables will be found useful in connection with the design of columns :—

TABLE LXII.

Form of section.	Area of section A.	Distance of y.	Moment of inertia about xx.	Moment of inertia about yy.
	$bt + dt_1$	$\frac{(b - t_1)t^2 + (d + t)^2 t_1}{2A}$	$\frac{bx_1^3 + t_1(d + t - x_1)^3}{3} - \frac{(b - t_1)(x_1 - t)^3}{3}$	$\frac{tb^3 + dt_1^3}{12}$
	$bt + dt_1 - tt_1$	$\frac{d}{2}$	$\frac{t_1 d^3 + (b - t_1)t^3}{12}$	$\frac{tb^3 + (d - t)t_1^3}{12}$
	$(2l - t)t$	$\frac{\frac{1}{2}\{l^3 - (l - t)(l^2 - t^2)\}}{(2l - t)t}$	$\frac{2\{x_1^4 - (x_1 - t)^4\}}{3} + \frac{t\{l - (2x_1 - \frac{1}{2}t)\}^3}{3}$	

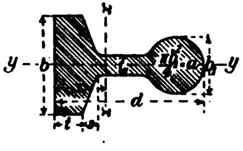
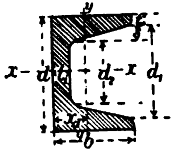
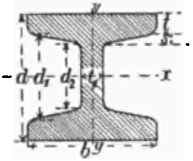
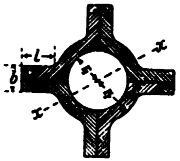
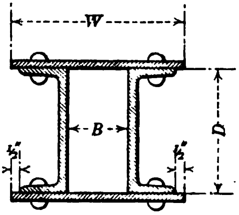
	$a + t_1(d - b_1)$ $+ (b - t_1)\left(t + \frac{s}{2}\right)$ $s = \frac{r}{2}(b - t_1)$	$\frac{a(2d - b_1) + t_1(d - b_1)^2}{2A}$ $+ \frac{(b - t_1)\left(t^2 + st + \frac{s^2}{3}\right)}{2A}$	$a\left\{\frac{b_1^2}{16} + \left(d - \frac{b_1}{2}\right)^2\right\}$ $+ \frac{t_1(d - b_1)^3}{3}$ $+ \frac{(t + s)^4 - t^4}{6r}$ $- Ax_1^2$	$\frac{\frac{2}{3}ab_1^2 + t_1^3(d - b_1 - t - s)}{12}$ $+ \frac{tb^3 + \frac{r}{8}(b^4 - t_1^4)}{12}$
 y x x d d₁ d₂ t t₁ s x₁ b y	$2bt + d_1t_1 + s(b - t_1)$ *or* $2bt + d_1t_1$ approx. $s = r(b - t_1)$	$\frac{b^2t + \frac{1}{2}d_1t_1 + \frac{s}{3}(b - t_1)(b + 2t_1)}{A}$ *or* $\frac{\frac{1}{2}\{b^2d - (b^2 - t^2)(d - 2t)\}}{2bt + d_1t_1}$	$\frac{bd^3 - \frac{1}{8r}(d_1^4 - d_2^4)}{12}$ *or* $\frac{bd^3 - (b - t)(d - 2t)^3}{12}$	$\frac{b^3\left(2t + \frac{s}{2}\right) + d_2t_1^3}{3}$ $- Ax_1^2$ *or* $\frac{2t_1(b - x_1)^3 + d - 2t_1}{3}$ $- (d - 2t_1)(x_1t)^3$
	$2bt + d_1t_1 + s(b - t_1)$ *or* $bd - (b - t_1)(d - 2t_1)$ approx. $s = \frac{r}{2}(b - t_1)$	$\frac{d}{2}$	$\frac{bd^3 - \frac{1}{4r}(d_1^4 - d_2^4)}{12}$ *or* $\frac{bd^3 - (b - t)(d - 2t)^3}{12}$	$\frac{2tb^3 + d_2t_1^3}{12}$ $+ \frac{r}{48}(b^4 - t_1^4)$ *or* $\frac{2tb^3 + (d - 2t)t_1^3}{12}$
	$\pi(r_2^2 - r_1^2) + 4bl$		$\frac{\pi(r_2^4 - r_1^4)}{4}$ $+ 2bl\left(r_2 + \frac{l}{2}\right)^2$	

TABLE LXIII.[1]

TABLES OF VALUES OF LEAST RADIUS OF GYRATION FOR VARIOUS SECTIONS OF COLUMNS.

TYPE 1.

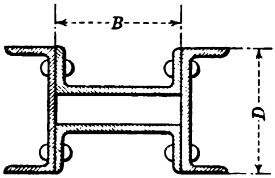


Columns of No. 2 channels { 8 in. × 3 in. × 3 in. × ½ in.
10 „ × 3 „ × 3 „ × ½ „
12 „ × 3½ „ × 3½ „ × ½ „

With No. 2 flange-plates . { ½ in. thick.
1 „
1½ „

								Remarks.
D . . =	in.	8	8	10	10	12	12	
B . . =	„	4	8	5	10	6	12	The least radius of gyration is in some of these around horizontal axis, and in others around vertical axis.
W . . =	„	11	15	12	17	14	20	
Least radius . =	„	3·06	3·7	3·4	4·56	3·99	5·44	When flange plates are ½ in. thick.
Sectional area =	sq. in.	24	28	27	32	32	38	
Least radius . =	in.	3·1	4·1	3·42	5·01	4·01	5·92	„ „ 1 „
Sectional area =	sq. in.	35	43	39	49	46	58	
Least radius . =	in.	3·12	4·42	3·44	5·13	4·02	6·05	„ „ 1½ „
Sectional area =	sq. in.	46	58	51	66	60	78	

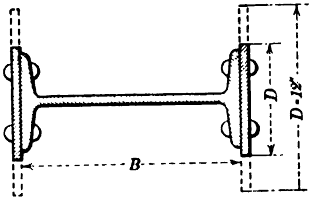
TYPE 2.



Columns of No. 4 channels { 8 in. × 3 in. × 3 in. × ½ in.
10 „ × 3 „ × 3 „ × ½ „
12 „ × 3½ „ × 3½ „ × ½ „

D . . =	in.	8	10	12	12	10	12
B . . =	„	8	10	12	8	8	10
Least radius =	„	2·52	3·26	3·94	4·18	3·32	4·13
Sectional area =	sq. in.	26	30	36	31	28	32

TYPE 3.



Columns of No. 1 joist section (Dorman, Long & Co.'s dimensions). { 20 in. × 7½ in. × 89 lb. per ft.
18 „ × 7 „ × 75 „
14 „ × 6 „ × 57 „
10 „ × 5 „ × 35 „

With No. 2 plates { D in. wide × ½ in. thick to 1 in. thick.

											Remarks.
D . . =	in.	6	9	12	7	12	8	12	8	12	
B . =	„	10	10	10	14	14	18	18	20	20	
Least radius =	„	1·33	1·92	2·63	1·53	2·44	1·72	2·37	1·72	2·30	With flange-plates ½ in. thick.
Sectional area =	sq. in.	16·28	19·28	22·28	23·75	28·75	30·06	34·06	34·2	38·2	
Least radius =	in.	1·44	2·16	2·95	1·65	2·78	1·85	2·7	1·85	2·63	With flange-plates 1 in. thick.
Sectional area =	sq. in.	22·28	28·28	34·28	30·75	40·75	38·06	46·06	42·2	50·2	

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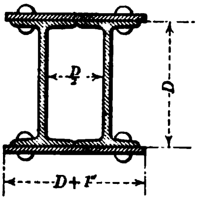
[1] Table LXIII. is taken from Moncrieff's papers.

TABLE LXIII.—*continued.*

TYPE 4.

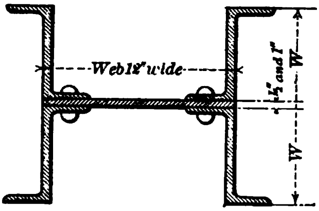


Columns of No. 2 joists (Dorman, Long & Co.'s sections) With distance apart centre to centre $= \frac{D}{2}$
8 in. × 4 in. × 25 lb. pei ft.
12 „ × 6 „ × 54 „
16 „ × 6 „ × 62 „
20 „ × 7½ „ × 89 „

With No. 2 flange plates D + 1 in. in width
½ in. thick.
1 „

D . . = in.	8	12	16	20	Remarks.
Least radius = in.	2·35	3·42	4·42	5·47	With flange plates ½ in. thick.
Sectional area = sq. in.	23·6	44·8	53·46	73·4	
Least radius = in.	2·42	3·5	4·56	5·61	„ 1 „
Sectional area = sq. in.	32·6	57·8	70·46	94·4	

TYPE 5.



Columns of No. 4 Z-bars
4 in. × 3 in. × 3 in. × ⅜ in.
6 „ × 3 „ × 3 „ × ⅜ „
8 „ × 3½ „ × 3½ „ × ⅜ „

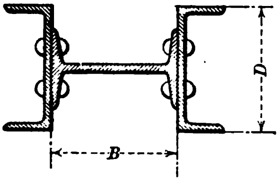
With web 12 in. wide ×
½ in. thick.
1 „

W = depth of one Z-bar.

W . . . . . = in.	4	6	8	Remarks.
Least radius . . = in.	2·29	3·42	4·62	With web plate ½ in. thick.
Sectional area . . = sq. in.	19·84	22·84	27·36	
Least radius . . = in.	2·17	3·21	4·36	„ 1 „
Sectional area . . = sq. in.	25·84	28·84	33·36	

NOTE.—A reduction in width of web plate slightly increases the least radius of gyration of these Z-bar columns.

TYPE 6.



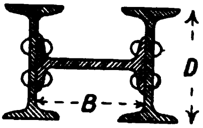
Columns of No. 2 channels
8 in. × 3 in. × 3 in. × ½ in.
10 „ × 3 „ × 3 „ × ½ „
12 „ × 3½ „ × 3½ „ × ½ „

With No. 1 joist section (Dorman, Long & Co.)
8 „ × 5 „ × 30 lb. per ft.
10 „ × 6 „ × 45 „
12 „ × 6 „ × 54 „
16 „ × 6 „ × 62 „
20 „ × 7½ „ × 89 „

D . . = in.	8	12	8	12	8	12	8	12	8	10	12
B . . = „	8	8	10	10	12	12	16	16	20	20	20
Least radius = „	2·40	3·63	2·28	3·42	2·21	3·30	2·12	3·19	2·09	2·49	3·01
Sectional area = sq. in.	21·81	26·81	26·23	31·23	28·9	33·9	31·23	36·23	39·2	41·2	44·2

TABLE LXIII.—*continued.*

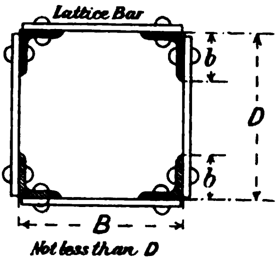
TYPE 7.



Columns of No. 3 joist sections (Dorman, Long & Co.'s dimensions).	8 in. × 4 in. × 19 lb. per ft. 12 „ × 5 „ × 59 „ 16 „ × 5 „ × 50 „ 20 „ × 7½ „ × 89 „

D . . = in.	8	12	16	20	8	12	16
B . . = „	8	12	16	20	12	16	20
Least radius = „	2·64	3·90	5·05	6·54	2·35	3·73	4·60
Sectional area = sq. in.	16·68	34·23	44·1	78·6	22·53	37·52	55·4

TYPE 8.



Column of No. 4 equal-sided angle-bars connected by lattice bars of flat or angle section (no plates).

D = least width over backs of angle-bars.
b = width of leg of angle-bar.
Least radius gyration approximately $= \frac{D}{2} - 0{\cdot}3\,b$.

CHAPTER XIII.

RIVETED JOINTS, PINS AND EYEBARS, JOINTS IN TIMBER.

In designing bridges, roofs and similar steel structures, we make use of certain rolled sections consisting of plates, angles, tees, Z-bars, flat and round bars, also occasionally specially rolled sections, connected together by means of rivets, pins, bolts and nuts, gibs, cotters, union-screws, etc. It is necessary to know how to arrange the lengths and sections of the various riveted joints and other connections, in the most convenient manner, not merely with a view to resist the various stresses developed in them, but also to reduce the cost of manufacture, the cost of carriage by land and sea to the site of the structure, and the subsequent erection. The designer must possess a list of sizes of plates and bars rolled by manufacturers, and all such plates and bars used in any built-up structures should be market sizes obtained from one of the many lists published, such as the following: H. J. Skelton & Co.'s *Handbook*, No. 10; *Structural Steel Pocket Companion*, Carnegie Steel Company. Lists of sizes, such as Messrs. Dorman, Long & Co.'s *The British Standard Sections and Specifications*. The list of rolled sections giving the properties, such as area, weight, moments of inertia and radii of gyration, are indispensable to the designer. For examples of riveted joints and connections in mild steel structures of various kinds the student is recommended to study *Notes on Construction*,[1] by H. Fidler, which should be used in conjunction with such handbooks as those mentioned.

Again, much valuable information on riveted work, bolts, nuts, screws, couplings, gib and cotter joints, is contained in Unwin's *Machine Design*, Part I.[2] The writer will not attempt to deal with this subject in a complete manner, as it would necessarily go over the ground covered by the various authorities mentioned, but the reader should carefully study the physical properties of the materials used in the rivets, plates and bars, etc., and should make himself familiar with such matters as relation of diameter of rivet to size of hole, length for various grips, the relative merits of drilling, instead of punching the holes, the effect of rhymering a hole previously punched, multiple punching and drilling, machine and hand riveting, field riveting, etc.

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[1] *Civil Engineering Series*, published by Longmans.

[2] Unwin's *Machine Design*, Part I., published by Longmans.

It is also necessary to know the relative cost of flat and round bars, angle tee and channel bars, plates and rivets, etc., in order to design economically.

There are some matters which will be referred to in this chapter more in detail as they are associated more especially with the design of structures.

Strength of Riveted Joints.—Consider a single riveted lap joint shown in Figs. 339 to 341. The usual method of expressing the various modes of failure is as follows:—

Failure may occur in such a joint in one of five ways—

1. By the crushing of the plate in front of the rivets.
2. By the rivets shearing.
3. By the rivet breaking through the plate at *cc*, Fig. 341.
4. By the plate tearing between the rivets across the section *ab*, Fig. 339.
5. By the rivet shearing out the plate in front of it, Fig. 341.

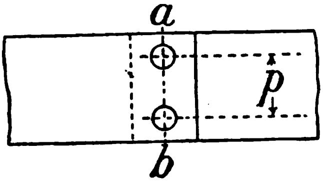


FIG. 339.

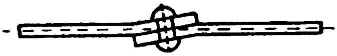


FIG. 340A.

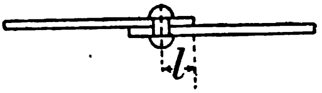


FIG. 340.

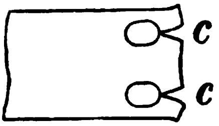


FIG. 341.

Let d = the diameter of a rivet, p = the pitch measured from centre to centre of the rivets, t = the thickness of the plate, l = the lap of the plate.

σ_t, σ'_s, σ_c = the tensile, shearing, and compressive strength of steel plate, also σ_s = the shearing strength of the rivet steel, then—

1. The resistance of the plate in front of a rivet is—

$$R = \sigma_c td.$$

2. The resistance to shearing of one rivet is—

$$R = \frac{\sigma_s \pi d^2}{4}.$$

3. The resistance to the plate breaking through at *cc*, Fig. 341, is found by equating the bending moment to the moment of resistance—

$$\frac{Rd}{8} = \frac{\left(l - \frac{d}{2}\right)^2 t\sigma_t}{6}$$

$$\therefore R = \frac{1}{3}\left(2l - d\right)^2 \frac{t\sigma_t}{d}.$$

4. The resistance of the plate to tearing between the rivets is—

$$R = \sigma_t(p-d)t.$$

5. The resistance of the plate in front of the rivet to shearing is—

$$R = 2\sigma_s'\left(l - \frac{d}{2}\right)t.$$

If the thickness of the plate is known, we have—

$$\sigma_c td = \frac{\sigma_s \pi d^2}{4} \therefore d = \frac{4t\sigma_c}{\pi\sigma_s}$$

$$\text{also } \sigma_t t(p - d) = \sigma_s\frac{\pi d^2}{4} \therefore p = d + \frac{\sigma_s \pi d^2}{\sigma_t 4t}$$

$$\text{again } 2\sigma_s'\left(l - \frac{d}{2}\right)t = \sigma_c td \therefore l = \frac{d}{2}\left(\frac{\sigma_c}{\sigma_s'} - 1\right).$$

The value of l is usually $1{\cdot}5d$, and $d = 1{\cdot}2\sqrt{t}$. In a similar manner the corresponding quantities may be determined for double riveted lap and butt joints, etc.

The stresses in actual riveted joints are very complex, and cannot be expressed in a simple manner, also the efficiency of a joint designed as above described may not be as calculated. The effect of punching, drilling and riveting, modify the actual results obtained, and the only reliable guide is by means of experiments. The stresses are not equally distributed over a group of rivets, so that we cannot state the precise amount carried by each rivet. If a wide plate is tested in tension up to the point of fracture, it will be seen that the middle portion of the plate has not elongated as much as the portions on each side of it. If holes are made in or near the centre of a specimen, a portion of the material in the front and rear of the holes will be relieved of stress, and the total stress will be more equally distributed over the sectional area across the holes; moreover the apparent strength of the material between the holes at right angles to the applied stress will be increased from 10 to 12 per cent. if the holes are drilled. In an ordinary test-piece tested in tension fracture occurs at the weakest section of a more or less considerable length of the bar. In the case of a drilled bar, it is not possible to select the weakest section of the plate, as the bar is compelled to break across the rivet holes, and moreover it is probable that the form, of the short portions of the material between the holes, restricts the necking and plastic flow to a much greater extent than in an ordinary test bar, so that the ultimate tensile strength is increased. If the holes are punched the resistance of the material about the hole is diminished, but a punched bar will show an increase in tensile strength across the holes of about 5 per cent. in excess of the plate tested in the ordinary way.

Generally in most forms of riveted joints, the resultant of the load does not pass through the middle of the plates, and there is a bending moment, equal to the stress multiplied by the eccentricity of loading, producing an unequal distribution of stress, and thus diminishing the strength of the joint.

The effect of this eccentricity is very great in the case of a lap joint shown in Fig. 340, as will be seen from the following calculation :—

Let P = the total load applied in tension.
„ t = the thickness of the plate.
„ b = the breadth of the plate.
„ σ_1 = the direct stress.[1]
„ σ_2 = the stress due to bending.
„ σ_t = the total stress.
„ I = the moment of inertia $= \frac{bt^3}{12}$.
„ $A = bt$.
„ y = the distance of extreme fibre from the neutral axis.
„ M = the maximum bending moment.

Then $M = Pt$; $\sigma_t = \sigma_1 \pm \sigma_2$; $\sigma_1 = \frac{P}{A} = \frac{P}{bt}$; $\sigma_2 = \frac{My}{I} = \frac{6P}{bt} = 6\sigma_1 \therefore \sigma_t = 7\sigma_1$ (tension)
$= 5\sigma_1$ (compression).

A mild steel lap joint illustrated in Figs. 339 and 340 was tested in the author's laboratory by Messrs. Barraclough and Gibson assisted by H. M. May, B.E., who elaborated the work for a thesis.

The object of the tests was to investigate by means of a number of Martin's mirror extensometers the distribution of the stresses over the joint as determined by the measured strains. The mirrors were distributed in such a manner as to show the strains occurring in any portion of the joint. The rivets were closed by hydraulic machinery, but the edges were not caulked. The joint illustrated in Figs. 339 and 340 was evidently subjected to a bending moment equal to Pt, if we neglect the effect of the frictional resistance, developed between the surfaces in contact, due to the clamping action of the rivets. Under these circumstances, we have shown that—

$$\sigma_t = 7\sigma_1 \text{ and } -5\sigma_1.$$

If, however, the frictional resistance is sufficiently developed to cause the two plates to act as one piece, then the bending moment is only $\frac{Pt}{2}$, and the stresses developed will be—

$$\sigma_t = 4\sigma_1 \text{ and } -2\sigma_1.$$

In any actual joint, the stresses developed must lie between $7\sigma_1$ and $4\sigma_1$ in tension, and between $-5\sigma_1$ and $-2\sigma_1$ in compression. The experiments on the joint illustrated in Figs. 339 and 340 showed that the maximum tensile stress was equal to $4\sigma_1$, so that the clamping action must have been effective. In regard to the modified joint illustrated in Fig. 340A which is frequently used, the stresses cannot be investigated in a simple manner, but the results of the experiments made in a similar manner to those on the joint (Fig. 340) showed that the maximum tensile stress was between $1{\cdot}75\sigma_1$ and $2\sigma_1$ in the joints tested. In the case of two boiler explosions which recently came under the author's notice, the shell was united with double riveted lap-joints, similar to the one illustrated in Fig. 340, and the plates fractured between the rivet holes along the whole length of the joint. The appearance of the fracture indicated that the process of destruction had been gradually developed until fracture ultimately occurred.

---

[1] The value of σ_1 in these experiments was about 3000 pounds per sq. in.

There appears to be no doubt that lap-joints such as the one illustrated in Fig. 340 are very unsatisfactory in consequence of the unequal distribution of the stresses developed under working conditions, which may in some portion of the joint exceed the elastic limit of the material. The effect of this overstraining must produce a gradual loss of ductility, similar to that observed under repeated stresses exceeding the elastic limit of the material. The result on the joint under consideration probably produced the crack which appears to have caused the explosion referred to. In tests of riveted joints up to the point of fracture, the plastic flow of the material in the most strained portions, tends to equalise the stresses, so that the results of such tests do not reveal the conditions existing under working stresses.

The eccentricity of loading unavoidable in the joints between lattice bars and chords of lattice bridges, also in the connections of the wind bracing, must produce results more or less similar to those referred to.

The ratio $\frac{\sigma_t}{\sigma_s}$ in riveted joints is not a constant. In steel plates with drilled holes it is equal to 1·26, 1·34, 1·36 in single-, double- and triple-riveted joints respectively. The pitch for a single-riveted joint with plates $\frac{1}{2}$ in. thick, and rivets $\frac{7}{8}$ in. diameter (actual size when closed), is—

$$p = \frac{\sigma_s \pi d^2}{\sigma_t 4t} + d = 1{\cdot}83 \text{ in.}$$

If the nominal diameter it $\frac{7}{8}$, the actual diameter would be larger; if it is taken at 0·95, the pitch would be 2·07 in.

In a double riveted joint with the same sizes—

$$p = \frac{\pi d^2 \sigma_s}{2t\sigma_t} + d = 2{\cdot}67 \text{ or } 3{\cdot}05 \text{ for diameters of } \tfrac{7}{8} \text{ and } 0{\cdot}95 \text{ respectively.}$$

The efficiency is equal to—

$$a\frac{p-d}{p}$$

where $a = 1 \cdot 0$ in drilled single-riveted, and $1 \cdot 06$ in drilled double-riveted joints.

The following tests of riveted joints are important:—

1. By a Committee of the Institution of Mechanical Engineers, 1880-1885 and 1888, published *Proc. Inst. Mech. E.*, also in *Engineering*, vols. xxix., xxxi., xxxix. and xl.

2. A series of tests made at the Watertown Arsenal, U.S. America.

3. A series made by Kirkaldy, published in *Engineering*, vol. xxix., p. 194.

4. A series of tests of riveted joints made for the Bureau of Steam Engineering, U.S. Navy Department.

5. Tests of riveted joints of iron plates, Low. Moor quality, and iron rivets closed by hand, made by Professor Tetmajer, published by F. Lohbauer, Zurich.

6. Tests of group joints suitable for girders made by the author, *Proc. University Engineering Society*, Sydney.

In regard to the tests 1; these were made under the supervision of Professor, now Sir A. B. W. Kennedy with steel plates and rivets, and he has proposed the following rules based on the results obtained:—

(*a*) *Single-riveted Lap-joints.*—The following values for diameter and pitch should be used wherever possible—

$$d = 2\tfrac{1}{3}t,\ p = 2\tfrac{3}{8}d.$$

The plate area is thus 71 per cent. of the rivet area. If smaller rivets are used, as is generally the case, he recommends the following formula—

$$p = a\frac{d^2}{t} + d.$$

The mean value of a, for plates 28-30 tons per sq. in. and rivets 22-24 tons per sq. in., is $0 \cdot 56$.

(*b*) *Double-riveted Lap-joints.*—Wherever possible—

$$d = 2\tfrac{1}{3}t,\ p = 3 \cdot 73d.$$

If this gives too large a rivet, then—

$$p = 1 \cdot 55\frac{d^2}{t} + d$$

for plates and rivets of the same resistance as given for (*a*).

(*c*) *Double-riveted Butt-joints.*—

$$d = 1 \cdot 8t,\ p = 4 \cdot 1d.$$

The results of this series of experiments would have been more valuable if wider strips had been used throughout.

2. Among the numerous tests made at the Watertown Arsenal, may be mentioned a series of 216 specimen single-riveted butt-joints; the thickness of the plates varied from $\frac{1}{4}$ to $\frac{3}{4}$ in., advancing by $\frac{1}{8}$ in., and the covers from $\frac{3}{16}$ to $\frac{7}{16}$ in. The material used in the plates was mild steel carefully annealed, the holes were drilled, and the rivets were of wrought-iron, ranging from $\frac{9}{16}$ to $\frac{13}{16}$ in. diameter, driven by machinery in holes $\frac{1}{16}$ in. larger than the nominal size of the rivets.

The elastic limit in the $\frac{3}{16}$ in. plate was 72·2 per cent. of the ultimate tensile strength, and in the $\frac{3}{4}$ in. plate it was only 53·3 per cent. The width of the plates varied from 9·75 to 15 in., and the mean tensile strength varied from 56,300 to 60,000 pounds per sq. in., with elongations of from 31 to 20 per cent. The tensile strength measured on the net section of the plates, $\frac{1}{4}$ in. thick, ranged from 61,960 to 81,230 pounds per sq. in., the compression on the bearing surface from 72,000 to 133,000 pounds per sq. in., the shearing stress on the rivets from 12,300 to 33,500 pounds per sq. in., and the efficiency ranged from 59 to 74 per cent. In the $\frac{3}{8}$ plates the diameter of the rivets ranged from $\frac{9}{16}$ to 1 in. and the pitch from $1\frac{5}{8}$ to $2\frac{7}{8}$ in. The results gave a tensile strength of 60,700 to 69,200 pounds per sq. in., and pressure on bearing area from 73,000 to 116,300 pounds per sq. in., also the shearing stress on the rivets ranged from 16,200 to 41,300 pounds per sq. in., and the efficiency from 62 to 75 per cent.

The tensile strength of the net section in this series varied over wide limits, but was always greater than that of the unperforated plate of the same quality. These tests show also that there is considerable latitude in the diameter and pitch of the rivets without changing the efficiency to any material extent. Thus in the $\frac{1}{4}$ in. plate joints—

$\frac{5}{8}$ in. rivets at $1\frac{7}{8}$ in. pitch gave 72·4 per cent. efficiency.
$\frac{3}{4}$,, ,, ,, $2\frac{1}{4}$,, ,, ,, 73·2 ,, ,, ,,
$\frac{7}{8}$,, ,, ,, $2\frac{3}{8}$,, ,, ,, 71·5 ,, ,, ,,
1 ,, ,, ,, $2\frac{1}{2}$,, ,, ,, 70·3 ,, ,, ,,
1 ,, ,, ,, $2\frac{7}{8}$,, ,, ,, 73·8 ,, ,, ,,

The actual areas of the net sections of plates and rivets did not give a constant ratio of one to the other, and for the same degree of efficiency it is evidently not necessary to have a fixed ratio between the rivet and plate areas. In the $\frac{1}{2}$-in. plate joints with $\frac{7}{8}$ rivets the efficiencies remained nearly constant over the range of pitches tested, the widest pitch $2\frac{1}{4}$ in. had 61·1 per cent. of the solid plate left, the narrowest pitch $1\frac{7}{8}$ had 53·4 per cent. of the solid plate left, the gain in strength in the net section of the latter compensated for the loss of metal.

The single-riveted butt-joints with $\frac{3}{4}$ in. plates and rivets $\frac{15}{16}$ in. to $1\frac{1}{4}$ in. diameter and pitches $2\frac{1}{2}$ to $2\frac{3}{8}$ gave the following results:—

Tensile strength of net section of plate	58,400 to 71,070 lb. per sq. in.
Compression on bearing surface of rivets	52,460 to 108,270 " " "
Shearing stress on rivets . . .	19,280 to 42,970 " " "
Efficiency	52·6 to 68·8 per cent.

4. In regard to the series of tests of riveted joints made for the Bureau of Steam Engineering U.S. Navy Department. These joints consisted of open hearth steel plates with drilled rivet holes and sheared edges. Both steel and wrought-iron rivets were used, driven by machinery.

The results of tests made on wide strips of the metal of about 7·5 sq. in. sectional area were as follows:—

Tensile strength in lbs. per sq. in. = 49,570 to 54,120.
Elastic limit " " " " = 24,000 to 28,950.
Elongation on 15 in. per cent. = 33·2 to 36·8.
Contraction of area per cent. = 39·8 to 49·4.

The details of these tests are given in the tables on pages 294, 295.

5. The late Prof. Tetmajer made eleven tests of riveted joints. The first four consisted of double cover butt-joints, with two rivets on each side, and were made to determine the distance from the axis of the rivet to the end of the plate. The results obtained show that with a single row of rivets on each side of the joint, having double covers and rivets in double shear, the distance from the centre of the hole to the end of the plate must be greater than 1·5 times the diameter of the rivet—

$$l > 1{\cdot}5d.$$

In the case of a joint having two or more rows of rivets on each side, the distance from the centre of the rivet hole to the end of the plate may be made 1·5 times the diameter of the rivet, with both single and double cover joints. The remaining seven tests consisted of single cover butt-joints, but with one, two and three plates riveted together with one to four rows of rivets, the joint in the plate being next to the cover, or with one plate between the joint and the cover. These seven tests have an important bearing on the design of joints in the flanges, or top and bottom members of girders, and the results show that wherever the joint is arranged unsymmetrically with reference to the thickness of the plates above and below it there is always a loss of strength due to bending, varying from 3 to 8 per cent. in the joints tested.

The experiments showed that it is sufficient to make the rivet area equal to the plate area, although for other reasons it is preferable to have an excess of rivet over plate area.

6. The experiments made by the author were for the purpose of determining the best proportions of rivet to plate area in various types

of group joints such as are used in girders. The plates and rivets were of mild steel, and the rivets were closed by hydraulic machinery. In the first seven tests the joints were designed to break in various ways, and did not represent well-proportioned joints, and they will not be described in detail.

The results obtained from these preliminary tests showed that in joints such as Fig. 345 the rivets are in single shear, and that in joints such as Figs. 342 and 343 the rivets in the central group are at least equivalent to twice single shear, but that the full value cannot be developed if the pressure on the bearing area is excessive. In joints of this character a certain amount of eccentricity is unavoidable, so that the stress developed is not exactly distributed over the net sectional area of the plates.

In all tests of riveted joints there are three well-defined periods which should be observed. First, the joint appears to be held by the rivets acting as clamps, and the frictional resistance developed prevents any movement of the joint except that due to the elastic stretch of the metal. In the second period the frictional resistance is overcome and the rivets act upon the holes like pins, producing what is called visible slip. In the third period the stretch of the joint appears to be due to the distortion of the rivets and holes and the plastic flow of the metal between the rivet holes.

The three following tests were made with two rows of rivets $\frac{1}{2}$ in. diameter and 2 in. pitch; the plates were $\frac{1}{4}$ in. thick, with covers $\frac{5}{16}$ in. thick. Fig. 342.

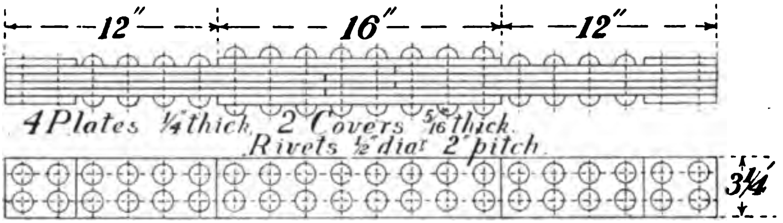
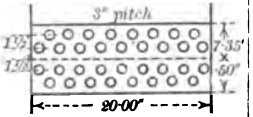
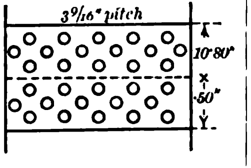
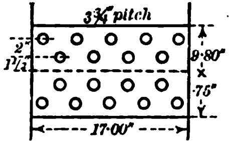
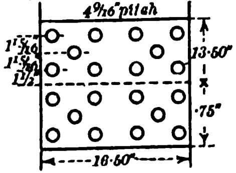


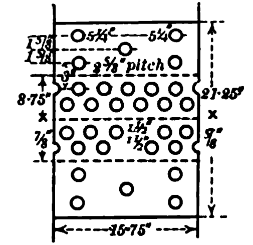
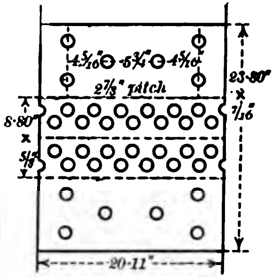
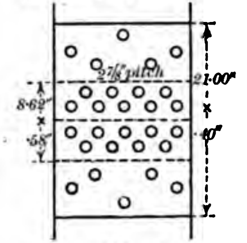
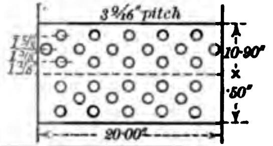
FIG. 342.

As might have been anticipated, this joint did not fail, but the plates broke across the net section outside the joint with a total load of 68 tons or 30·2 tons per sq. in. The material in these tests gave the following results in testing specimen cut from the plate to standard form :—

Tensile strength = 28·7 tons per sq. in.
Yield point = 21·4 ,, ,,
Elongation on 8 in. = 1·6 in.
,, ,, 4 in. = 1·1 ,,
Contraction of area = 46 per cent.

TABLE LXIV.

No. of test.	Style of joint.	Nominal thickness of plate.	Size and kind of rivets and holes.	Sectional area of plate.		Bearing surface of rivets.	Shearing area of rivets.	Tensile strength of plate, pounds per sq. in.	Maximum stress on joint per sq. in.				Efficiency of joint.
				Gross.	Net.				Tension on gross section of plate.	Tension on net section of plate.	Compression on bearing surface of rivets.	Shearing on rivets.	
		In.		Sq. in.	Sq. in.	Sq. in.	Sq. in.		Lbs.	Lbs.	Lbs.	Lbs.	Per cent.
910	 3" pitch 1 3/4" 1 3/8" 7·35" X ·50" Y 20·00"	$\frac{5}{8}$	$\frac{3}{4}$ in. steel rivets $\frac{25}{32}$ in. drilled holes	13·22	10·12	6·72	12·46	53,710	42,860	55,990	84,320	45,470	79·8
911		$\frac{5}{8}$		13·08	10·01	6·64	12·46	53,710	40,960	53,530	80,690	43,000	76·2
912		$\frac{5}{8}$		12·41	9·50	6·30	12·46	53,710	42,720	55,800	84,140	42,540	79·5
913		$\frac{5}{8}$	$\frac{3}{4}$ in. steel rivets $\frac{25}{32}$ in. drilled holes	12·81	10·31	8·01	15·34	53,710	43,460	54,040	69,560	36,320	80·9
914		$\frac{5}{8}$		12·69	10·21	7·93	15·34	53,710	40,390	50,200	64,630	33,410	75·2
915		$\frac{5}{8}$		12·691	10·21	7·94	15·34	53,710	44,290	55,050	70,790	36,640	82·5
916	 3 3/4" pitch 2" 1 1/2" 9·80" X ·75" Y 17·00"	$\frac{7}{8}$	1 in. steel rivets $1\frac{1}{16}$ in. drilled holes	14·671	10·09	8·25	15·96	51,190	35,180	51,550	62,560	32,340	68·7
917		$\frac{7}{8}$		14·646	10·07	8·23	15·96	51,190	36,190	52,640	64,410	33,210	70·7
918		$\frac{7}{8}$		14·705	10·11	8·27	15·96	51,190	35,780	52,050	63,630	32,970	69·9
919	 4 9/16" pitch 1 15/16" 1 15/16" 1 1/2" 13·60" X ·75" Y 16·60"	$\frac{7}{8}$	1 in. steel rivets $1\frac{1}{16}$ in. drilled holes	14·322	10·63	10·15	19·51	51,190	37,910	51,080	53,500	27,830	74·1
920		$\frac{7}{8}$		14·438	10·72	10·23	19·51	51,190	38,400	51,720	54,190	28,420	75·0
921		$\frac{7}{8}$		14·256	10·58	10·10	19·51	51,190	37,950	51,130	53,560	27,780	74·1

922	 5¼" 5¼" 2⅝" pitch 1⅜" 8·75" ⅞" 1½" 21·25" 9" 15·75"	$\frac{7}{8}$	3 plates $1\frac{1}{16}$ in. iron rivets $1\frac{1}{8}$ in. drilled holes	13·781	10·93	16·30	28·00	51,190	40,810	$\underline{51{,}460}$	34,500	20,090	79·7
923		$\frac{7}{8}$	2 plates $1\frac{1}{8}$ in. iron rivets $1\frac{1}{4}$ in. drilled holes	13·852	10·985	16·38	28·00	51,190	41,770	$\underline{52{,}640}$	35,300	20,650	81·5
924		$\frac{7}{8}$		13·741	10·90	16·26	28·00	51,190	40,120	$\underline{50{,}580}$	33,910	19,690	78·4
925	 4 5/16" 5¾" 4 5/16" 2⅞" pitch 8·80" 5/8" 23·80" 7/16" 20·11"	$\frac{5}{8}$	3 plates 1 in. iron rivets $1\frac{1}{16}$ in. drilled holes	13·252	10·537	14·045	30·41	53,710	42,820	$\underline{53{,}860}$	40,410	18,660	79·7
926		$\frac{5}{8}$	2 plates $1\frac{1}{8}$ in. iron rivets $1\frac{1}{4}$ in. drilled holes	12·776	10·154	13·548	30·41	53,710	45,300	57,000	42,720	19,030	84·3
927		$\frac{5}{8}$		12·81	10·186	13·576	30·41	53,710	46,070	$\underline{57{,}940}$	43,470	19,410	85·8
928		$\frac{9}{16}$	$\frac{15}{16}$ in. iron rivets 1 in. drilled holes	8·287	6·820	8·057	18·06	—	42,250	51,330	43,450	19,380	—
929		$\frac{9}{16}$		8·211	6·755	7·994	18·06	—	40,800	$\underline{49{,}590}$	41,910	18,550	—
930		$\frac{9}{16}$		8·28	6·81	8·05	18·06	—	40,720	$\underline{49{,}520}$	41,890	18,670	—
931		$\frac{5}{8}$	$\frac{3}{4}$ in. iron rivets $\frac{25}{32}$ in. drilled holes	12·36	9·94	7·73	15·34	53,710	43,600	54,220	69,720	35,130	81·2
932		$\frac{5}{8}$		12·93	10·40	8·08	15·34	53,710	43,260	$\underline{53{,}780}$	69,220	$\underline{36{,}460}$	80·5
933		$\frac{5}{8}$		12·98	10·44	8·11	15·34	53,710	43,000	$\underline{53{,}460}$	68,820	$\underline{36{,}390}$	80·1

(The figures underlined indicate the manner of failure.)

Specimen cut from the plates and milled for a length of 24 in. to the same width as the joint 3·25 in., and drilled in a similar manner with two rows of rivets ½ in. diameter and 2 in. pitch, gave the following results :—

Tensile strength = 31·5 tons per sq. in.
Yield point = 22·4 ,, ,,
Elongation on 20 in. = 1·187 in.
,, ,, 10 in. = 0·7 ,,
Contraction of area = 31·5 per cent.

The increase in tensile strength and reduction in ductility in the perforated over the unperforated plate is normal.

The ½ in. rivets were tested in double shear and gave a mean value reduced to single shear of 22·4 tons per sq. in. Hence the joint shown in Fig. 342 may be considered as fairly well proportioned for ordinary girder work ; the excess of rivet over plate area is on the safe side. A second test was made on a joint similar to the last, except in the central group of rivets, which has only two instead of four rivets ; also the covers were only ¼ in. thick. Fig. 343.

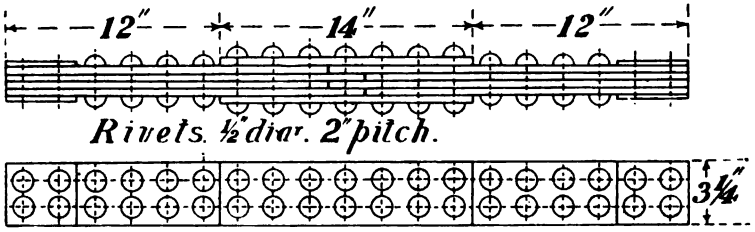


Fig. 343.

The rivet area in central group = 0·3927 sq. in.
,, ,, end groups = 1·1781 ,,
The plate area net = 2·2500 ,,
,, ,, gross = 3·2500 ,,
The cover plates net = 1·1250 ,,

The joint failed by tearing the two cover plates, and at the same time the rivets in the central group cut through the plates with a total load of 59 tons. The stress necessary to break the cover plates based upon the foregoing test under similar conditions is—

$$1·125 \times 30·2 = 34 \text{ tons.}$$

The load upon the rivets in the central group was therefore—

$$59 - 34 = 25 \text{ tons.}$$

The stress upon the rivets was—

$$\frac{25}{0·3927} = 63·6 \text{ tons per sq. in.}$$

—equivalent to $\frac{63{\cdot}4}{22{\cdot}4} = 2{\cdot}8$ times the value of a rivet in single shear. The pressure on the bearing area, assuming that it is equally distributed over the four plates, was—

$$\frac{25}{2 \times 2 \times \frac{1}{2} \times \frac{1}{4}} = 50 \text{ tons per sq. in.}$$

This excessive pressure caused the rivets to cut into the plates, as shown in Fig. 344.

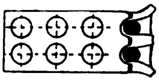


FIG. 344.

The last two experiments furnish data for the design of group joints of this type. Fig. 345 shows a stepped joint which broke across the net section of the plates outside the joint, as it was impossible to adjust the rivet to the plate area.

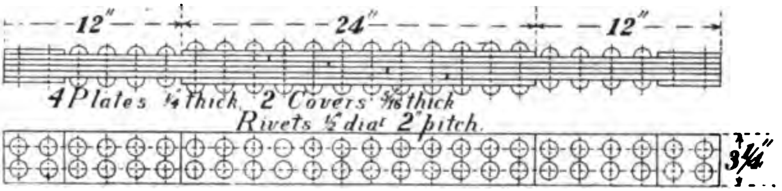


FIG. 345.

The total load applied was exactly the same as in the joint shown in Fig. 342—68 tons; so that the two joints may be compared as to quantities of material.

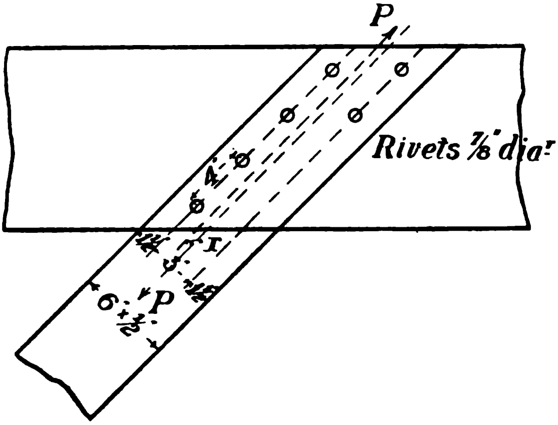


FIG. 346.

The effect of eccentric loading is shown in the following example. Fig. 346, representing the attachment of a flat bar, in which the rivets are unsymmetrical with reference to the axis of the bar 6 in. wide by $\frac{1}{2}$ in. thick, by means of six rivets each $\frac{7}{8}$ in. diameter.

The centre of gravity of the group of rivets is $x = \frac{1}{2}$ in. from the axis of the bar, so that the bending moment is Px. The force P produces uniform stress of intensity σ_1 thus—

$\sigma_1 = \frac{P}{A}$ where A denotes the net area of the bar $(6-2)\frac{1}{2} = 2$ sq. in.; if $P = 8$ tons, then $\sigma_1 = 4$ tons per sq. in.

The moment $Px = \frac{8}{2} = 4$ inch tons, and the moment of resistance $= \frac{6 \times 6}{6 \times 2}\sigma_2 = 3\sigma_2$ where $\sigma_2 =$ the extreme fibre stress.

$$\therefore 3\sigma_2 = 4, \text{ and } \sigma_2 = 1 \cdot 4 \text{ about.}$$

Hence the total stress in tension due to the direct stress, and that due to bending is $\sigma_1 \pm \sigma_2$. The maximum tensile stress is therefore $4 + 1\cdot4 = 5\cdot4$ tons per sq. in., and the maximum compressive strength is $4 - 1\cdot4 = 2\cdot6$ tons per sq. in.

This unequal distribution of stress is avoided by arranging the rivets symmetrically, or by the use of pin connections as in some truss bridges. As far as possible the rivets should be arranged symmetrically, in the joint between the bottom member of a lattice girder, or truss, and the bars forming the web members with reference to the axis of the bars, and the line passing through the centre of gravity of the cross-section of the member. It is not always possible to arrange the rivets perfectly symmetrically especially in the attachment of wind bracing bars, but these should have the rivets symmetrically arranged with reference to the axis of the bar. In any case, there is an unequal distribution of stress similar to that in a lap-joint in all connections of the types referred to in the manner discussed on page 288.

In any riveted joint the maximum efficiency is attained when the strength of the rivets is just equal to the strength of the plate. Fig. 347 shows an arrangement in which the net area of the plates joined is the greatest possible as well as the best arrangement of securing a uniformly distributed stress over the rivets and plates. The plate

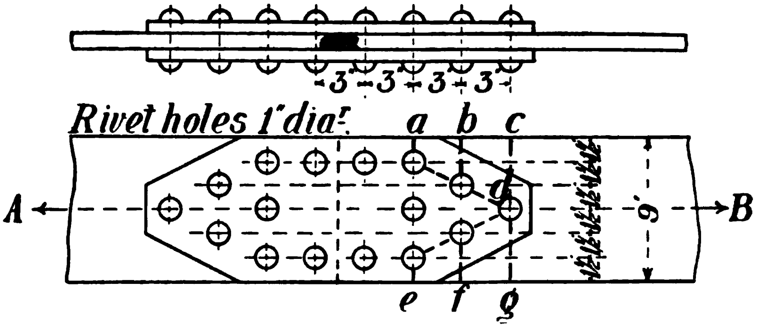


FIG. 347.

may fail by tearing apart between the rivet holes on the broken lines *ade*, *bdf*, or *cdg*. On the line *ade* failure would occur partly by direct tension on a section at right angles to the axis, and partly by longitudinal shear parallel to the axis. The shearing stress is about three-fourths of the tensile, so that the net area on the broken line *ade* is $9 - 5 = 4$ in. of width for net tension, and $(6 - 2)2 = 8$ in. in length for longitudinal shearing, equivalent to 6 in. in tension. The equivalent net width is therefore 10 in. On the line *bdf* it is 9 in., and on the line *cdg*, 8 in. The weakest section is therefore 8 in. net width. If the pitch longitudinally were made 2 in. instead of 3 in., the relative net widths would be more nearly equal along the broken lines. The net area of the cover plates should be equal at least to the net area of the plate joined, and it is usually made somewhat greater. If there is another plate between the one shown and the bottom cover, the moment of the areas of each cover about the centre of the joint ought to be equal, from which it follows that the covers should not be of equal thickness. If the two plates are each $\frac{1}{2}$ in. thick, and t_1 and t_2 denote the thickness of the covers, $t_1 + t_2 = \frac{1}{2}$ and $\frac{t_1}{2} = t_2$, hence the top cover should be $t_1 = \frac{1}{3}$, and the bottom cover $t_2 = \frac{1}{6}$. In a pile of plates with a joint at the centre, or a joint consisting of a single plate, the covers are usually made of equal thickness.

Ratio of Rivet Area to Plate Area.—The tensile strength of steel plates may be taken at 65,000 pounds per sq. in. and the shearing strength of steel rivets at 47,000 pounds per sq. in., or a ratio of 1·4 to 1, but in order to allow for imperfections in workmanship, and the variations from other causes, the ratio for average cases should be increased by about 10 per cent. or the rivet area in single shear should be 1·54 times the plate area. Cooper specifies that the holes deducted from the plates to obtain net sections in tension members should be assumed to be $\frac{1}{8}$ in. larger than the undriven rivets. The rivet holes are actually $\frac{1}{16}$ in. greater than the nominal size of the rivet.[1]

Although the ratio of rivet to plate area is the most important consideration in the strength of riveted joints, still under some circumstances the joint may fail in other ways even when this ratio is as above stated. The plates or rivets may be crushed due to excessive pressure, or the plates may break away in front of the rivets. The pressure on the bearing area in tested riveted joints is sometimes as great as 110,000 pounds per sq. in. without causing apparent injury, but the safe bearing pressure allowed by experienced designers is from 15,000 to 22,000 pounds per sq. in.; the latter value would be used for a dead load stress.

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[1] This practice ensures an excess of rivet over plate area, and the ratio of 1·54, used in the design of group joints in girders, provides a further excess of rivet strength, sufficient to meet all contingencies.

Let b = the breadth of a plate after deducting the diameter of each rivet hole plus $\frac{1}{8}$.

t = the thickness of the plate.

a = the area of a rivet.

n = the number of rivets required on each side of the joint.

Then neglecting pressure on bearing area, we have—

$$1{\cdot}54bt = na, \qquad \text{single shear} \therefore n = \frac{1{\cdot}54bt}{a}$$

$$1{\cdot}54bt = 1{\cdot}75na, \text{double shear} \therefore n = \frac{0{\cdot}88bt}{a}.$$

The pressure on the bearing area should be calculated after the number of rivets has been determined, and if it exceeds the allowed safe bearing pressure, the number or size of the rivets must be increased. Generally it will be found that double shear joints with thin plates produce excessive bearing pressures, but that in single shear joints it will not be necessary to increase the rivet area for the purpose of reducing the pressure upon the bearing area.

DESIGN OF RIVETED JOINTS IN GIRDERS.

Figs. 348 to 353 represent various group joints in the tension flange of girders, and the arrangement of the plates, angles and rivets illustrate the method of designing such joints. The thickness of the cover or splice plates in Figs. 348 to 351 and Fig. 353 must be made slightly thicker than that of the plates covered, as two joints in the plates come in the same plane vertically. In Fig. 352 the thickness of each of the two cover plates should be the same as that of each plate covered since they are all ½ in. thick, as the lever arm with reference to the joint in the lower plate on the left and the bottom cover is the same as the lever arm with reference to the joint in the upper plate on the right and the top cover. If the plates vary in thickness it will depend upon their position, in so far as they affect the length of the lever arms of the covers or splice plates in regard to the joint. If each of the four plates in Fig. 352 is ½ in. thick the cover plates should also be of the same thickness. In regard to the joints shown in Figs. 348 to 351 and also in Fig. 353, it follows from the foregoing reasoning that the cover plates should be about one-fifth thicker than one of the four plates covered, so that if the plates are ½ in. thick the covers should be ⅝ in. thick. In Figs. 348 to 351 the angle bars uniting the web to the flange plates are stopped against the inner cover plate, and a recess is cut out of the web plate of the same length as the cover plate, and with a depth equal to the thickness of the cover plate. The angle covers, or wrappers, rest upon the top of the inner cover plate, and are arranged to extend beyond it a sufficient distance

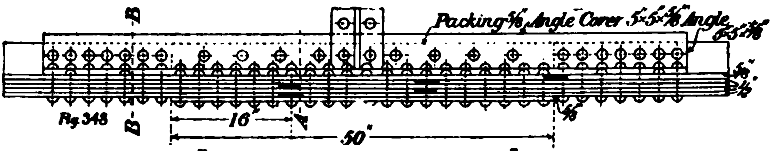


Fig. 348

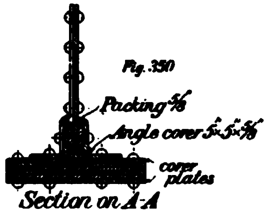


Fig. 350

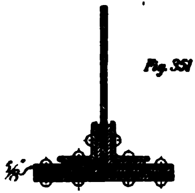


Fig. 351

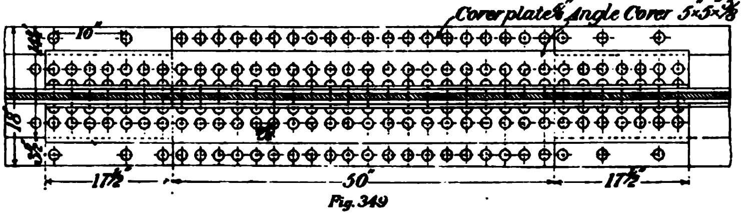


Fig. 349

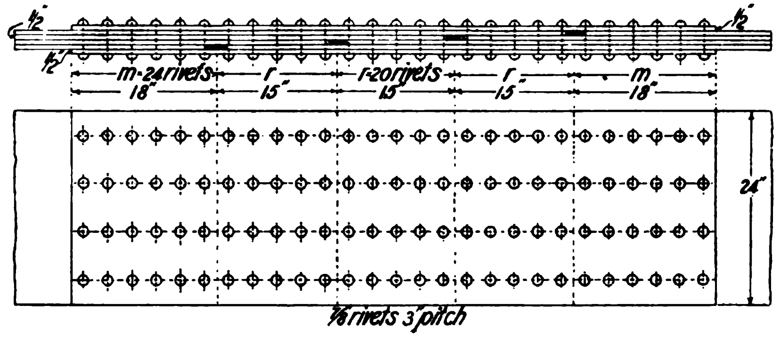


FIG. 352.

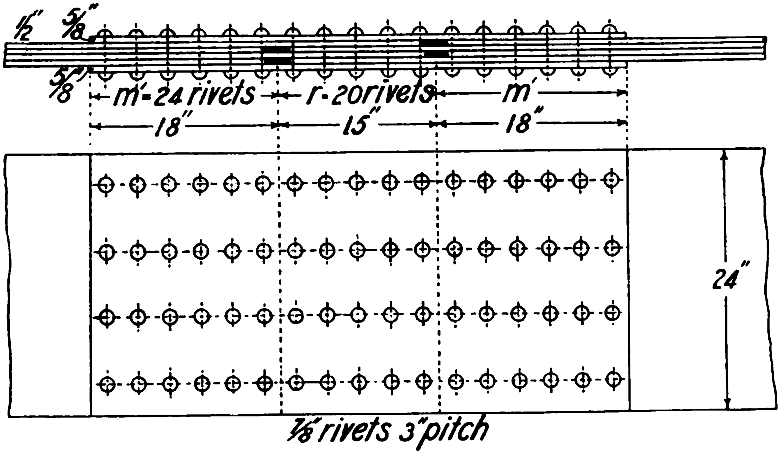


FIG. 353.

to provide the necessary rivet area. The rivet area in the portion of the angle covers extending beyond the inner cover plate must be at least 1·54 times the effective area of the two main angle bars. The effective area of the two main angle bars 5 × 5 × $\frac{5}{8}$ in. tension is 9·3 sq. in., so that the rivet area necessary is 14·3, or say 15 sq. in. The number of rivets $\frac{7}{8}$ in. in diameter required is 25 in single shear. The joint (Figs. 348 to 351) shows seven rivets on one side in double shear and fourteen rivets in single shear, which are equivalent to twenty-six rivets in single shear. This joint was suggested to the author by Mr. J. H. Frazer, of the Government Railway Department, Victoria.

In the group joints illustrated in Figs. 352 and 353 failure may occur in three ways:—

1. By the shearing of all the rivets.
2. By the shearing of the rivets in the central groups or group and the tearing of the cover plates at the same time.
3. By the rivets cutting into the plates, or the plates into the rivets, from insufficiency of bearing area.

In regard to the first mode of failure: Let n = the total number of rivets in a joint, p = the number of plates, b = the effective width of a plate, t = thickness of a plate; a and d represent the area and diameter of a rivet respectively. The number of rivets to resist shearing in joint Fig. 352—

$$n = \frac{1{\cdot}54bpt}{a}.$$

If m = the number of rivets in the end groups Fig. 352, and m' = the number in Fig. 353, r = the number of rivets in each of the central groups Fig. 352 and in the central group Fig. 353, then for the first mode of failure (Fig. 352)—

$$n = \frac{1{\cdot}54bpt}{a} = 2m + (p - 1)r \quad . \quad . \quad . \quad (1)$$

In Fig. 353 the rivets in the end groups are more nearly in double than in single shear, and it will be safe to assume that they are equivalent to one and a half times their value in single shear.

For the second mode of failure: In Fig. 352, since each cover is made equal to t, the cover plate area = $2bt$.

The number of rivets in the central groups Fig. 352 = $(p - 1)r$
„ „ „ „ group Fig. 353 = r.

The rivet areas are = $(p - 1)ra$, Fig. 352, and = ra Fig. 353. The equivalent plate area Fig. 352 = $\frac{(p - 1)ra}{1{\cdot}54}$

$$\therefore \; 2bt + \frac{(p - 1)ra}{1{\cdot}54} = pbt \quad . \quad . \quad . \quad (2)$$

The rivet area in the central group Fig. 353 is equivalent to at least twice that of the rivets in Fig. 352, since they would be sheared at three sections. The author made a number of experiments to determine the value of these rivets, and the results show that they are at least equal to twice that of single shear, hence—

$$2bt' + \frac{2ra}{1{\cdot}54} = pbt \quad . \quad . \quad . \quad . \quad (3)$$

where t' = the thickness of the cover plates.

Equations (2) and (3) may be solved to find r; then the value of r is substituted in equation (1) to find m for Fig. 352, or m' in Fig. 353: $m = \frac{3m'}{2}$. The pressure on the bearing area is most conveniently examined after the joint has been designed to resist the first and second modes of failure, and if it exceeds the limit assigned, 18,000 pounds per sq. in., the diameter or number of the rivets must be increased.

A group joint similar to either Fig. 352 or Fig. 353 may now be designed: Let there be four plates in each case 24 in. wide and $\frac{1}{2}$ in. thick; also four rows of rivets $\frac{7}{8}$ in. diameter and 3 in. pitch. In Fig. 352 the rivets are all in single shear, and—

$$n = \frac{1{\cdot}54 \times 4 \times 10}{0{\cdot}6} = 102{\cdot}6 = 104 \text{ rivets, about.}$$

The cover plate area $= 2 \times 10 = 20$ sq. in. The rivet area in each of the central groups is $0{\cdot}6r$, and in the three central groups $3 \times 0{\cdot}6r = 1{\cdot}8r$. The equivalent plate area is—

$$\frac{1{\cdot}8r}{1{\cdot}54} = 1{\cdot}17r.$$

The equation (2) becomes—

$$20 + 1{\cdot}17r = 40 \quad \therefore\ r = 17, \text{ or say } 18,$$

but since there are four rows of rivets, r must be a multiple of 4, and is therefore made equal to 20.

For the first mode of failure, equation (1), we have: $2m + 3r = 104$, or $m = 25$ if $r = 18$, and $m = 22$ if $r = 20$; here also the number must be a multiple of four, and m is made equal to 24. The total number of rivets in the joint, Fig. 352, is 108.

In regard to the pressure on the bearing area—

The bearing area in the three central groups is $60 \times \frac{7}{8} \times \frac{1}{2} = 26{\cdot}25$ sq. in. The stress on these rivets must be according to the second mode of failure one-half of the total stress in the four plates; hence, the ratio of tearing to bearing area is—

$$\frac{20}{26{\cdot}25} = 0{\cdot}8 \text{ about.}$$

The pressure on the bearing area is only 0·8 times the stress on the plates, so that the joint shown in Fig. 352 is satisfactory.

In regard to the joint shown in Fig. 353—

$$\text{The equivalent plate area} = \frac{2r \times 0{\cdot}6}{1{\cdot}54} = 0{\cdot}78r.$$

The net area of two cover plates, each $\frac{5}{8}$ in. thick, is 25 sq. in.

For the second mode of failure—

$$25 + 0{\cdot}78r = 40, \text{ or } r = 20, \text{ about.}$$

For the first mode of failure we should require 106 rivets if they were all in single shear, but since the rivets in the central group are equivalent to double their value in single shear, and in the end groups one and a half times their value in single shear, the arrangement shown in Fig. 353 may be adopted, which provides: $r = 20$, equivalent to 40 rivets in single shear; $2m = 48$, equivalent to 72 rivets in single shear.

The maximum pressure on the bearing area in terms of the resistance per sq. in. of plate in tension may be found in the following manner:—

The bearing area provided by the 20 rivets in the central group on two plates $\frac{1}{2}$ in. thick is—

$$2 \times 20 \times \frac{7}{8} \times \frac{1}{2} = 17{\cdot}5 \text{ sq. in.}$$

The bearing area provided by the 24 rivets in each of the end groups is 21 sq. in. The stress on the rivets in the central group after the cover plates have been torn apart is expressed in terms of the resistance of the plate—

$$40 - 25 = 15 \text{ sq. in.}$$

Or the stress is equivalent to the resistance of 15 sq. in. of plate.

The pressure on the bearing area in the central group, and the resistance of the plate in tension is expressed by the ratio—

$$\frac{17{\cdot}5}{15} = 1{\cdot}166.$$

The pressure is 1·166 times the intensity of stress upon the plates.

In a similar manner the pressure on the rivets in the end groups is equivalent to—

$$40 - 20 = 20 \text{ sq. in.}$$

and

$$\frac{21}{20} = 1{\cdot}05.$$

The pressure on the bearing area of the rivets in the end groups is 1·05 times the stress upon the plates.

The joint shown in Fig. 353 is therefore satisfactory.

The following table has been prepared by the American Bridge Company, and will be found useful in connection with the design of riveted joints:—

TABLE LXV.

SHEARING AND BEARING VALUE OF RIVETS IN POUNDS.

Diameter of rivet, in.		Area in sq. in.	Single shear at 11,000 lbs.	Bearing value for different thicknesses of plate in in. at 22,000 pounds per sq. in.												
Fraction.	Decimal.			$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{1}{2}$	$\frac{9}{16}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{3}{4}$	$\frac{13}{16}$	$\frac{7}{8}$	$\frac{15}{16}$	1
$\frac{3}{8}$	·375	·1104	1210	2060	2580	3090										
$\frac{1}{2}$	·500	·1963	2160	2750	3440	4130	4820	5,500								
$\frac{5}{8}$	·625	·3068	3370	3440	4300	5160	6020	6,880	7,740	8,600						
$\frac{3}{4}$	·750	·4418	4860	4130	5160	6190	7220	8,250	9,280	10,320	11,340	12,380				
$\frac{7}{8}$	·875	·6013	6610	4810	6020	7220	8430	9,630	10,840	12,040	13,240	14,440	15,640	16,840	18,050	
1	1·000	·7854	8640	5500	6880	8250	9630	11,000	12,380	13,750	15,130	16,500	17,880	19,250	20,630	22,000

All bearing values above or to right of upper zigzag lines are greater than double shear.
Values below or to left of lower zigzag lines are less than single shear.

Eye-bars.—Riveted joints and connections are characteristic of British and European practice in bridge building, but in America the main connections of tension members, both in the lower chord and diagonal braces, are made by means of pins. The tension members consist of eye-bars which are designed so that the weakest section is in the bar itself.

Fig. 354A represents a form of hammer-forged eye-bar, which, according to Prof. Burr, has stood the test of long American practice. It may be drawn as follows:—

Let w denote the width of the bar, and r the radius of the pin; then make $hb = 0{\cdot}87w$, and abc a semicircle with o_1 as centre and radius $= r + 0{\cdot}66w$; $oo_1 = oo_2$, and the curves ef and dg are portions

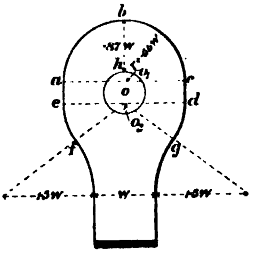
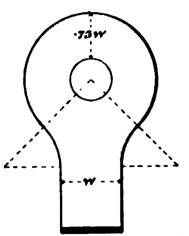
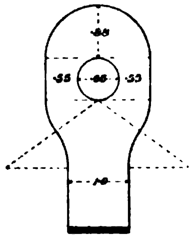


FIG. 354A.



Hydraulic forged.

FIG. 354B.



Kirkaldie.

FIG. 354C.

of a semicircle of the same radius as abc. The curve joining ef and dg to the bar is made so as to give a large amount of metal in the vicinity of o_2f and o_2g.

Fig. 354C shows a form of hammered head based upon experiments made by Mr. Kirkaldy. The head is frequently made circular, as shown in Fig. 354B, in which case the eye-bar is forged by hydraulic machinery. If it is desired to thicken the head so as to limit the mean intensity of pressure on the bearing area of the pin, let p denote the safe intensity of pressure; T, the maximum intensity of tensile stress in the bar; t, the thickness of the bar; and t_1, the thickness of the head; then—

$$wt\text{T} = 2rpt_1, \text{ and } t_1 = \frac{wt\text{T}}{2rp}.$$

Size of Pins.—The diameter of the pin should be at least sufficient to resist crushing and to limit the intensity of the pressure on the bearing area of the eye-bar to a safe amount. Let d denote the diameter of the pin, then $tdp = \text{T}tw$.

$$\therefore\ d = \frac{\text{T}}{p}w.$$

The ratio of the safe working intensity of tensile stress, T, to the compressive stress, p, is about $\frac{3}{4}$, so that the smallest permissible diameter of the pin is—

$$d = \tfrac{3}{4}w.$$

The smallest diameter usually taken is—

$$d = \tfrac{4}{5}w.$$

In Mr. Kirkaldy's eye-bar the diameter of the pin, $d = 0{\cdot}66w$, is the smallest which will break the bar. The diameter of the pin is usually greater than this value, as the pull of the eye-bars in the bottom chord and diagonals tend to bend it.

The maximum bending moment on the pin must be found by resolving all the forces which act upon it both horizontally and vertically; thus if M_h denote the sum of the moments of the horizontal forces, and M_v the sum of the moments of the vertical forces, then—

$$M = \sqrt{M_h^2 + M_v^2}$$

$$M = \frac{fI}{y}; \text{ which becomes}$$

$$M = \frac{\pi f d^3}{32} \text{ since the section is circular.}$$

$$\therefore d = \sqrt[3]{\frac{32M}{\pi f}}.$$

The value of f is usually taken as 15,000 lbs. per sq. in. for iron and 20,000 lbs. per sq. in. for steel; so that we have—

$$d = 0{\cdot}089 \sqrt[3]{M} \text{ (iron).}$$

$$d = 0{\cdot}081 \sqrt[3]{M} \text{ (steel).}$$

It is usual to select the proper diameter of the pin, when the bending moment M is known, from tables calculated by the two last formulæ.

The bending moment M may be calculated with sufficient accuracy on the assumptions—

(*a*) That the pressure applied to any pin has its centre at the centre of the surface of contact.

(*b*) That the centre of pressure is the centre of such a surface as will reduce the intensity of pressure on the bearing area to its safe maximum value, which for bridge steel is about 12,000 lbs. per sq. in., or about one-half of what has been found to be the limit for rivets. The pin does not fit the hole, as there is generally a clearance of $\frac{1}{50}$ of an inch, and the surface over which the pressure is actually exerted increases with the stress in the bar from a line contact to a surface more or less approaching dt.

Referring to the joint shown in Fig. 355, and denoting the distances between the centres of the eye-bars by x_1, x_2, x_3, etc., and the maximum tensile stress in the various bars by T_a, T_b, T_c, etc., we have—

About the centre of eye-bar D . $T_a(x_1)$
" " " B . $T_a(x_1 + x_2) - T_d(x_2)$
" " " E . $T_a(x_1 + x_2 + x_3) - T_d(x_2 + x_3) + T_b(x_3)$
" " " C . $T_a(x_1 + x_2 + x_3 + x_4) - T_d(x_2 + x_3 + x_4) + T_b(x_3 + x_4) - T_e(x_4)$, etc.

The stress in the counterbrace R does not usually act when the maximum stresses occur in the pin, and may therefore be neglected. The stress in the diagonal brace G must be resolved both horizontally and vertically, and the moments of the horizontal component about the centre of the post F added to the sum of the moments produced by the eye-bars in the bottom chord about the same point; the result M_h must be added to the moment of the vertical component M_v about the same point, thus—

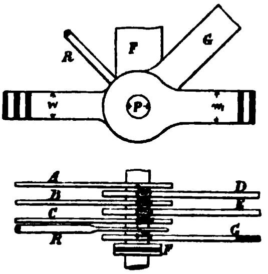


Fig. 355.

$$M = \sqrt{M_v^2 + M_h^2}.$$

The resultant may also be found thus: Set off Mx to scale representing the moment of the eye-bars in the bottom chord about the centre of the column F, and Mg also to scale in a direction perpendicular to that of the diagonal brace G, representing the moment of the diagonal brace about F. Complete the triangle; then M will represent to scale the magnitude of the greatest moment to which the pin is subjected, and its direction is the axis of the moment. The maximum bending moment in the pins of the lower chord will generally occur when the bridge is loaded completely, but in the upper chord when the bridge is partially loaded.

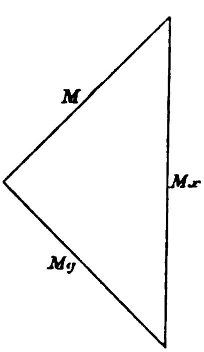


Fig. 356.

The eye-bars should be grouped so that they are symmetrical with reference to the axis of the lower chord, and so that the positive and negative bending moment stand to balance each other as much as possible. It will be observed that if the eye-bar is thickened in order to reduce the pressure on the bearing area, the length of the lever arm is increased with a corresponding increase in the bending moment, necessitating a larger pin. When the size of the pin is sufficient to resist the maximum bending moment, and for bearing area, it will be amply sufficient to resist shearing.

The pressure on the bearing area is increased in consequence of the practice of allowing $\frac{1}{32}$ to $\frac{1}{16}$ of an inch between the pin-hole and the pin for convenience in erecting. Mr. Theodore Cooper[1] has recently called attention to the behaviour of eye-bars when subjected to stresses of 12,000, 16,000, 20,000, 24,000 and 28,000 lbs. per sq. in. It was found by careful tape measurements on the eye-bars, from out to out of the pin-holes, that a permanent set occurred at 12,000 lbs. per sq. in., and that the elongation increased with each increase of stress. The amount of the pin clearance did not modify the results especially. Lines were ruled on the eye-bars parallel and normal to the length of the bars forming 1 in. squares, and the distortions of these squares indicated the nature and the relative intensities of the stresses. It was clearly proved by these experiments that the elastic limit of the eye-bars as a whole was reached long before that of the bar proper, and this appears to have been due to the high compressive stress in the eye-bar head at the back of the pin, mainly caused by the intensity of bearing pressure on the pin-hole. Fig. 357 shows the nature of these strains, the line XY is the transverse line through the centre of the pin-hole. The curved lines tangent to the elongated pin-hole are the forms taken by the straight lines tangential to the original pin-hole before the straining of the bars. The upper part of the pin-hole is held to the form and diameter of the pin, and has elongated more than the lower half. The diameter of the lower half has decreased by the transverse closing-in of the material under the pull. At the top of the pin the metal of the head is decreased in depth by the compression. The transverse dimensions across the eye and neck are reduced by the flow of the metal under tension. The portions denoted by H, H_1, had elongated so little that the stresses could not have exceeded the elastic limit, although the bar had been stressed up to rupture. It was considered how a series of such bars would pull together, and Mr. Cooper concluded that for long bars of about 50 ft. the stresses would be fairly equally divided, but that for short bars the distorsion of the heads would influence the result disproportionately.

The fact that a bar which had an elastic limit of 35,000 lbs. per sq. in. should show a permanent set when tested as an eye-bar at 12,000 lbs. per sq. in. was known to Bauschinger, and was shown in tests previously made at the Watertown Arsenal, but Mr. Cooper has investigated the effect of this on the Quebec bridge of 1800 ft. span, where eye-bars were used 15 in. by $1\frac{1}{4}$ to $2\frac{1}{16}$ in. thick, and from 50 to 58 ft. long. The pins used in this bridge were 12 in. in diameter and from 8 to 10 ft. long. The maximum joint has 58 bars on one pin.

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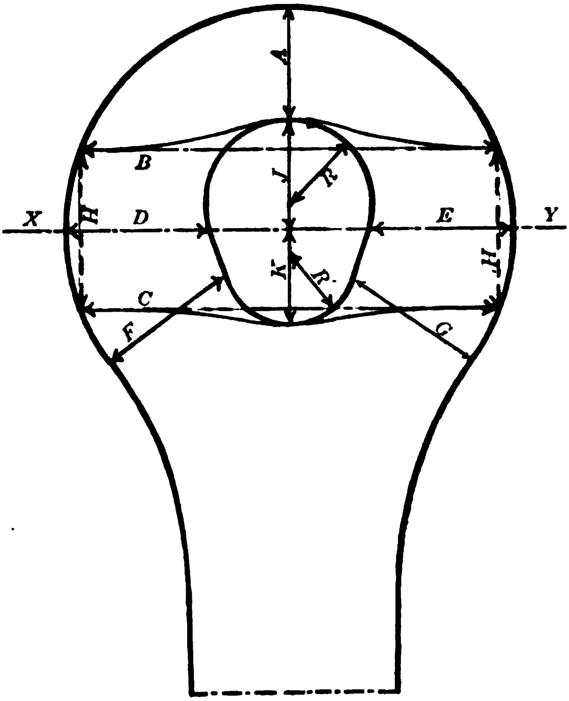
[1] "New Facts about Eye-bars," *Proc. Am. Soc. C.E.*, vol. xxxii., No. 1, Jan. 1906.

Deformation of heads of eye-bars under rupture.



X-Y is the line through centre of the original pin-hole.

FIG. 357.

Pins are usually hammer forged, or rolled from ingots or blooms of open hearth steel, the size of the ingots or bloom being about 50 per cent. greater than the finished size of the pin. After the pins have been manufactured to the correct diameter they should be heated slowly and uniformly to a dark red heat, and left to cool slowly for a period not less than twenty-four hours. Test-pieces cut near the surface of the annealed pins of not less than one-half of a square inch in cross-sectional area, should give an ultimate tensile strength of not less than 60,000 or more than 70,000 lbs. per sq. in., an elastic limit of not less than one-half of the ultimate strength, an elongation of not less than 20 per cent. in 8 in., or a contraction of area of not less than 40 per cent. Test-pieces cut near the centre of the pins should comply with the above requirements, except that a fall of 5000 pounds per sq. in. will be allowed in the ultimate strength and a fall of 5 per cent. in the contraction of area. The fractures should be free from granulation. In designing pin-connected joints, a table of maximum moments on the pins for a definite fibre stress, giving diameters and areas, such as in the American Bridge Company's specification, will be found useful.

Joints in Timber.—Fig. 358 represents the plan and elevation of a tension joint in timber, consisting of two cover-plates notched into the beam at each end and bolted as shown.

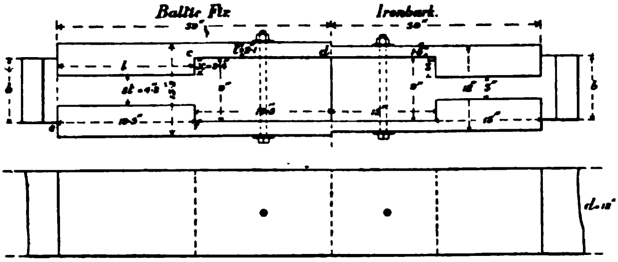


FIG. 358.

The beam is weakened to the extent represented by the reduction of area due to the notching, and the length of the notches on either side of the joint must be such that the shearing resistance will be equal to the tensile resistance of the beam at the reduced section, neglecting the resistance of the bolts.

Clearly the thickness of the two cover-plates in the centre must equal the reduced thickness of the beam if the same material is used for both, which is indicated in the figure by t and $2t$. Again, the shearing areas cd and ef must be equal. The depth of the two notches must be sufficient to offer a crushing resistance equal to the tensile resistance of the reduced area of the beam.

Assuming the following data for the intensity of working stress in pounds per square inch for Baltic fir and ironbark, we may dimension the right-hand side of Fig. 358 for the latter material, and the left-hand for the former.

Tension for Baltic fir, 1300, for ironbark, 4000.
Compression ,, ,, 1100, ,, ,, 2000.
Shearing ,, ,, 140, ,, ,, 400.
Bearing ,, ,, 1300, ,, ,, 4000.

Let b = breadth of beam.
d = depth ,,
l = length of notch.
x = depth of notch cut in beam.
t = thickness of cover at the centre.
$2t$ = thickness of reduced section of beam.

If we take the depth of the beam 12 in., and the breadth 9 in., we have for Baltic fir—

$$2t = b - 2x$$
$$\therefore\ 2t = 9 - 2x, \text{ and } t = 4{\cdot}5 - x$$
$$\therefore\ t = \tfrac{11}{13}x = 4{\cdot}5 - x$$
$$\text{since } 1300t = 1100x$$
$$\therefore\ x = 2{\cdot}4 \text{ in., and } t = 2{\cdot}1 \text{ in., and } 1300t = 140l$$
$$\therefore\ t = \tfrac{14}{130}l = 2{\cdot}1$$
$$\therefore\ l = 19{\cdot}5 \text{ in.}$$

For ironbark—

$$t = 4{\cdot}5 - x \text{ as before}$$
$$\text{but } 4000t = 2000x$$
$$..\ t = \frac{x}{2} = 4{\cdot}5 - x$$
$$\therefore\ x = 3 \text{ in., and } t = 1{\cdot}5 \text{ in., } 4000t = 400l$$
$$\therefore\ t = \frac{l}{10} = 1{\cdot}5$$
$$\therefore\ l = 15 \text{ in.}$$

In both cases the strength of the joint is much less than the beam, in consequence of the reduction of area due to notching.

The Baltic-fir joint has an efficiency of 47 per cent. nearly.

The ironbark „ „ „ „ 34 „ „

The joints shown in Fig. 359 would naturally suggest themselves in which the crushing and shearing resistances are the same as in Fig. 358.

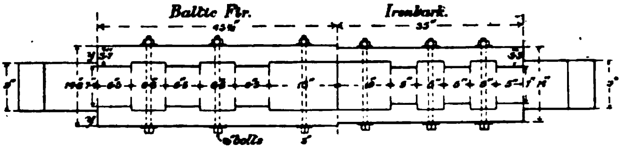


FIG. 359.

Neglecting the resistance of the bolts, the dimensions may be as shown—

Efficiency of Baltic-fir joint = 82 per cent.
„ „ ironbark joint = 77·7 „

If the bolts are considered we should not have to cut the notches so deeply, or the efficiency would not be increased, but the length of the notches and spaces between them would not require to be so great.

The objection to this and the preceding joint is that the timber is notched at right angles to its length, rendering it likely to fracture at yy with a lower stress than the reduced area would resist under favourable circumstances, so that the efficiencies given above are probably too great.

Fig. 360 shows a modification of the joint shown in Fig. 357, which may be taken to have the efficiencies above stated, as it is not so likely to fail in consequence of sharp corners.

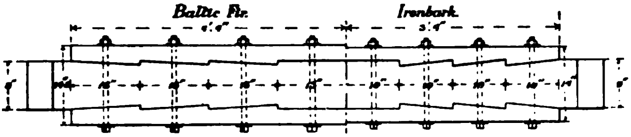


FIG. 360.

Fig. 361 illustrates a form of joint used in Australia in the bottom chords of timber truss bridges, which are frequently constructed of ironbark timber. The form of the truss is indicated in Fig. 217, Chapter VIII., and the joint under consideration occurs in the third and sixth bays. The chord is constructed with four beams, each

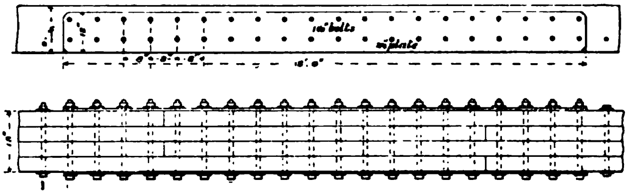


FIG. 361.

4½ in. wide by 14 in. deep, bolted together with bolts 1 in. in diameter, arranged in a zigzag form, 10 bolts being used in each bay. The apex loads and stresses have been considered in Chapter VIII.

The joint is designed for a maximum tensile stress of 109·1 tons, and the chord for this stress, as well as for the transverse stresses due to the loads discharged by the cross-beams between the panel-points. The dimensions and arrangements of the bolts and cover plates shown in Fig. 361 has been arrived at after a series of experiments on full-sized joints of similar construction. In all cases the joints experimented upon failed by the bending of the bolts and the crushing of the fibres of the timber on the bearing area, and the following safe working stresses per bolt were deduced :—

Bolts 1⅛ in. in diameter—1·85 ton stress in single and 3·7 tons in double shear.

Bolts 1 in. in diameter—1·34 ton stress in single and 2·68 tons in double shear.

In this way the strength of the joint shown in Figs. 359 and 360 may be estimated by considering the strength of the bolts, thus :—

16 bolts $1\frac{1}{8}$ in. in diameter in single shear = 16 × 1·85
= 29·6 tons
24 bolts $1\frac{1}{8}$ in. in diameter in triple shear = 3 × 24 × 1·85
= 133·2 tons.

The total strength of the joint, neglecting the ⅜ in. cover plates, is 162·8 tons. The joint is therefore excessively strong.

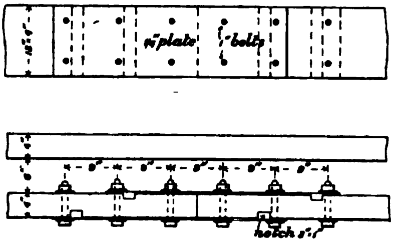


FIG. 362.

Fig. 362 illustrates another form of joint used in the bottom chords of trusses indicated in outline in Fig. 218. In this case the chord is made up of two beams, each 12 in. deep by 4 in. wide, spaced 6 in. apart.

There are two strips on each cover plate, 2 in. long by 1 in. wide by 12 in. deep, which fit into corresponding notches cut in the timber. The maximum tensile stress in this case is 53·04 tons.

The strength of this joint may be considered with reference to the bolts on one side of it, and the crushing strength of the timber at the notches, thus:—

Strength of two notches in half-chord = 2 × 1 in. × 12 × 2000
= 48,000 lbs. = 21·4 tons
4 bolts in double shear 1 in. diameter = 10·72 „
2 „ single „ 1 „ „ = 2·68 „
Total strength of half-chord = 34·80 tons
Total „ „ whole „ = 69·6 „

This joint is therefore abundantly safe, since the maximum tensile stress is only 53·04 tons. There is an advantage in regard to the number of notches; two surfaces may be arranged to bear equally, whereas with a greater number it is practically impossible to obtain the same bearing pressure on each notch. The shearing resistance along the fibre should be at least equal to the crushing resistance, but this does not arise in the joint under consideration, as the area of the timber between the joint and the notch multiplied by the shearing strength is obviously greater than the area subjected to crushing multiplied by the crushing strength.

Again, the area of the timber in tension is only slightly reduced by the notch and the bolts, and need not be considered, as it is obviously excessive.

CHAPTER XIV.

WIND PRESSURE ON STRUCTURES. STRESSES DUE TO CURVATURE ON STRUCTURES. THE EFFECT OF BRAKES.

THE published information on this subject which may be consulted for more complete information includes the following :—

Papers by Mr. C. B. Bender and Prof. Gaudard, published in *Proc. Inst. C.E.*, vol lxix., p. 80, with discussions by Sir B. Baker and others.

"Experiments by Sir B. Baker at the site of the Forth Bridge," published in *Engineering*, 28th February, 1890.

"Experiments by Lieut. O. T. Crosby," published in *Engineering*, 30th May, 6th June and 13th June, 1890.

Paper by Mr. C. Shaler Smith, published in *Trans. Am. Soc. C.E.*, vol. x., p. 139.

Paper by F. E. Nipher, "A Method of Measuring the Pressure at any Point on a Structure due to Wind Blowing against the Structure," *Trans. Academy of Science*, St. Louis, vol. iii., No. 1, 14th January, 1898.

Paper by Dr. Stanton of the National Physical Laboratory, "On the Resistance of Plane Surfaces in a Uniform Current of Air," published in *Proc. Inst. C.E.*, vol. clvi.

Report of the National Physical Laboratory, 1907. "The Estimation of the Wind Pressure on Structures," by Dr. Stanton and Mr. Bairstow.

When a horizontal force of wind strikes an inclined surface it causes a normal pressure upon that surface, the intensity of which is expressed by the following formula deduced from experiments made by Hutton for gunnery purposes.

Let P denote the horizontal pressure on an inclined surface, such as a roof inclined at an angle of θ to the horizon, and let N denote the normal pressure, then—

$$N = P\,(\sin\theta)^{1 \cdot 84 \cos\theta - 1}$$

Duchemin found that—

$$N = \frac{2\sin\theta}{1 + \sin^2\theta}$$

The following table gives the values of N for different values of θ, when P = 40 pounds per sq. ft. :—

TABLE LXVI.

Angle of roof, θ.	Normal pressure, N.		Angle of roof, θ.	Normal pressure. N.	
	Hutton.	Duchemin.		Hutton.	Duchemin.
5°	5·1	6·89	50°	38·1	38·64
10°	9·7	13·59	60°	40·0	39·74
20°	18·4	24·24	70°	40·0	39·91
30°	26·5	32·00	80°	40·5	40·00
40°	33·3	36·40	90°	40·0	40·00

The experiments of Sir B. Baker show that the average pressure over a large surface is less than over a small surface. The large gauge used for these experiments was 20 ft. long by 15 ft. deep, exposing 300 sq. ft. of surface; a circular gauge 18 in. in diameter was also arranged in the centre of the large gauge.

The small gauge consisted of a circular plate of 1·5 sq. ft. area.

The revolving gauge also exposed a surface of 1·5 sq. ft., but was so arranged that it turned round and always faced the direction of the strongest wind.

The large and small fixed gauges were arranged to face the direction from which the strongest winds blow. Some of the results are recorded in the following table:—

TABLE LXVII.

Year.	Month and day.	Pressure in pounds per square foot.				Direction of the wind.
		Revolving gauge.	Small fixed gauge.	Large fixed gauge.	Circular gauge 18″ in diameter in centre of large gauge.	
1884	Oct. 27	29	23	18	...	S.W.
,,	,, 28	26	29	19	...	S.W.
1885	March 20	30	25	17	...	W.
,,	Dec. 4	25	27	19	...	W.
1886	March 31	26	31	19	...	S.W.
1887	Feb. 4	26	41	15	...	S.W.
1888	Jan. 5	27	16	7	...	S.E.
,,	Nov. 17	35	41	27	...	W.
1889	,, 2	27	34	12	...	S.W.
1890	Jan. 19	27	28	16	...	S.W.
,,	,, 21	26	38	15	...	W.
,,	,, 22	27	24	18	23½	S.W. by W.

The relation between the velocity of the wind and the pressure on a surface may be calculated if we assume that a column of moving air acts upon a relatively large surface so that the direction of the moving air is completely changed. Let W denote the weight of air directed against a surface in a given time; A, the area of the surface; w, the weight of a cubic foot of air; v, the velocity in feet per second, and g the acceleration due to gravity (32·2 ft. per second), then—

$$W = wAv,\ P = \frac{Wv}{g} = \frac{wAv^2}{g}.$$

If $A = 1$ sq. ft., $w = 0{\cdot}0807$ lbs., $V =$ velocity in miles per hour, then it follows that—

$$P = p = 0{\cdot}0054V^2.$$

The value of the constant will vary with the temperature and pressure, but the expression cannot be applied to the case of wind acting upon a structure, as it neglects the cushioning of the air on the wind side, and the vacuum effect on the lee side.

The value of the constant is best determined by experiment, and the best information is to be derived from the experiments made at the National Physical Laboratory in Great Britain described in a paper published in *Proc. Inst. C.E.*, vol. clvi., "On the Resistance of Plane Surfaces in a Uniform Current of Air," by Dr. Stanton, which is well worth a careful study. Dr. Stanton found that for square or circular plates of small dimensions that—

$$P = 0{\cdot}0027V^2.$$

Dines in *Proc. Royal Society*, vol. xlviii., p. 252, that—

$$P = 0{\cdot}0029V^2.$$

Froude, *ibid.*, vol. xlviii., p. 253, that—

$$P = 0{\cdot}00366V^2.$$

Hazen in 1886 found—

$$P = 0{\cdot}0034V^2.$$

Langley, at the observatory of Allegheny, Pa., found for velocities of 10 to 70 miles an hour that—

$$P = 0{\cdot}0032V^2 \text{ to } P = 0{\cdot}0039V^2.$$

In the present imperfect state of our knowledge we may use the expression—

$$P = 0{\cdot}004V^2.$$

The investigation of this subject at the National Physical Laboratory suggested in the report for 1907, "An Estimation of the Relation between the Mean Square of the Velocity over Large Areas, ranging from 100 to 100,000 square feet, and the Velocity at Some Definite Point in the Area," when completed will supply most valuable data on this subject. Sir B. Baker stated in the discussion on Dr. Stanton's paper that windows in houses were rarely smashed in during a high gale, and he found that the resistance of ordinary windows was about 30 pounds per sq. ft., front pressure, to which should be added the leeward pressure. By calculating the overturning moment of railway carriages and trucks, it is clear that the average wind pressure on the surface exposed in such cases rarely exceeds 30 pounds per sq. ft., as only in very exposed situations and under exceptionally strong winds have railway vehicles been overturned. Since 30 pounds per sq. ft. would overturn most of the rolling stock on existing railways, it is not necessary to estimate a greater pressure on the exposed surface of a bridge and train, although a greater pressure may be allowed on the bridge itself without the train. The correct estimate of the exposed surface is very important, and the amount of shelter afforded by the truss or girder on the wind side to that on the leeward side. As the pressure on structures due to wind is greater in proportion to the height of the structure above the ground, tall chimneys, piers of high viaducts, Eiffel towers, etc., may be assumed to be required to resist about 40 pounds for the first 100 ft., and from 50 to 60 pounds above 100 ft. in height. In braced piers, towers and bridges, in order to provide completely for the effect of the strongest winds on the windward and leeward sides, it is safe to double the area exposed on the windward side, and to assume a maximum pressure of 50 pounds per sq. ft. The British Board of Trade Rule requires that bridges in exposed situations should be calculated for 56 pounds per sq. ft. on double the exposed area on the windward side. The importance of the subject in connection with the design of large structures may be realised by a consideration of Sir B. Baker's estimate of the stresses in the Forth bridge, which are as follows :—

Stresses due to dead load = 2228 tons.
,, ,, ,, live ,, = 1022 ,,
,, ,, ,, wind ,, = 2920 ,,
Total = 6170 ,,

In Cooper's *Specification for Steel Railroad Bridges*, 1901, the following statements occur :—

"(*a*) To provide for wind stresses and vibrations from high-speed trains, the top lateral bracing in deck-bridges, and the bottom lateral bracing in through-bridges, shall be proportioned to resist a lateral force of 600 pounds for each foot of span; 450 pounds of this to be treated as a moving load, and as acting on a train of cars, at a line 6 feet above the base of rail.

"(*b*) The bottom lateral bracing in deck-bridges, and the top lateral bracing in through-bridges, shall be proportioned to resist a lateral force of 150 pounds for each foot for spans up to 300 ft., and 10 pounds additional for each additional 30 ft.

"(*c*) In trestle towers the bracing and columns shall be proportioned to resist the following lateral forces, in addition to the stresses from live and dead loads—

"(1) With either one track loaded with cars only, or with both tracks loaded with maximum train load, the lateral forces specified in (*a*), and a lateral force of 100 pounds for each vertical lineal foot of the trestle bents; or

"(2) With both tracks unloaded, a lateral force of 500 pounds for each longitudinal foot of the structure, acting at the centre of the girders, and a lateral force of 200 pounds for each vertical lineal foot of the trestle bents.

"(*d*) For determining the requisite anchorage for a loaded structure, the train shall be assumed to weigh 800 pounds per lineal foot."

In the American Bridge Co.'s *Specifications for Steel Railroad Bridges*, 1900, the following occurs:—

"The wind pressure shall be assumed as acting in either direction horizontally.

"First. At 30 pounds per sq. ft. on the exposed areas of the surface of all trusses and the floors, as seen in elevation, in addition to a train of 10 ft. average height, beginning 2 ft. 6 in. above the base of rail, moving across the bridge.

"Second. At 50 pounds per sq. ft. on the exposed surface of all trusses and the floor system.

"The greatest result shall be assumed in proportioning the parts."

In the *Specification* of the same company for highway bridges, 1901, the first clause is modified in so far as it relates to a train, and the following is substituted:—

"At 30 pounds per sq. ft. on the exposed surface of all trusses and the floor, as seen in elevation, in addition to a horizontal live load of 150 pounds per lineal foot of the span moving across the bridge."

The method of determining the stresses in a bridge due to wind is illustrated in the design of a plate web girder, Plate I., and also in a Pratt-truss railway bridge, Plates II. and II.*a*. Details of the design of the top and bottom lateral systems of wind bracing, and of the sway and portal bracing, are sufficiently illustrated in Plates I., II., II.*a*, II.*b*, III. and III.*a*.

Stresses due to Curved Structures.—When a train passes over a bridge built on a curve a force is developed acting along the radius of the curve tending to press the train against the outer rail. This centrifugal effect may be calculated from the general formula—

$$C = \frac{Wv^2}{gr}.$$

The force acts as a live load about 5 ft. above the base of the rail, and is proportional to the square of the velocity, and inversely proportional to the radius of the curve. The speed is less on sharp curves, and therefore the centrifugal force should be assumed as the greatest possible on the particular curve. Mr. Cooper specifies a speed of $(60 - 3D)$ miles per hour where $D = \frac{5730}{R}$. D denotes the degree of curvature, and R the radius in feet. If C denotes the centrifugal force, and V the velocity in miles per hour, then it can be shown that—

$$C = 0{\cdot}00001167WV^2D.$$

The American Bridge Co., 1900, specify the following formula :—

$$C = 0{\cdot}03WD.$$

The coefficient 0·03 is reduced 0·001 for every degree of curvature above 5 degrees.

Effect of Brakes.—When brakes are applied to the wheels of a train passing over a bridge stresses are developed which may have to be specially provided for, as in the case of the cross-bracing in viaducts. The N.Y.C. & H.R.R.R. Co.'s 1904 specifications require that "the stresses produced in the bracing of the trestle towers, in any members of the trusses, or in the attachment of the girders or trusses to their bearings, by the greatest tractive force of the engines, or by suddenly stopping the trains on any part of the bridge, must be provided for. The coefficient of friction of the wheels on the rails will be assumed at 0·20. In all structures provision shall be made for trains going in either direction and on any track. The stresses produced in the trestle legs from this force need not be considered."

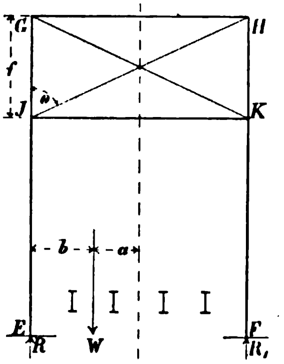


Fig. 363.

Professor Waddell[1] gives an investigation for determining the stresses in the vertical sway bracing of a double-track bridge, which may be briefly described as follows :—

When only one track of a double-track bridge without vertical sway bracing is covered by the moving load, the trusses are loaded unequally, the load being divided according to the law of the lever.

If vertical sway bracing is used, we may assume that the load is equally divided between the trusses, the difference between the loads being taken up by the sway bracing, thus (Fig. 363)—

---

[1] *Memoirs of the Tôkiô Daigaku*, by Professor J. A. L. Waddell.

Let R and R_1 = the reactions produced by a load on one track.
W = the load.
G = the load transferred by the bracing; then—

$$G = R - \frac{W}{2} = W\left(\frac{2a+b}{2(a+b)}\right) - \frac{W}{2} = \frac{Wa}{2(a+b)}.$$

The stress in the vibration rod JH is—

$$G \sec\theta = \frac{Wa \sec\theta}{2(a+b)}.$$

The stress in JK is—

$$\frac{Wa}{2(a+b)} \times \frac{2(a+b)}{f} = \frac{Wa}{f}.$$

The stress in GH can be shown to be zero.

The bending moment at K is—

$$M' = \frac{Wa}{2(a+b)}\{2(a+b)\} = Wa.$$

This bending moment does not exist at the same time as the maximum stress in the post HF, and need not generally be considered.

The bending moment produced in the long vertical posts about the centre of the trusses in double-line railway bridges will in general require the section of the post to be increased in area beyond that found for the ordinary live and dead loads. It should be remembered that this bending moment has been calculated on the assumption that the reactions R and R_1 have been exactly equalised by the bracing; again, no account has been taken of the fact that the vertical post acts as a beam fixed at the ends in resisting the bending moment at K. Taking these matters into consideration, it follows that the actual bending moment is probably not greater than $\frac{Wa}{3}$.

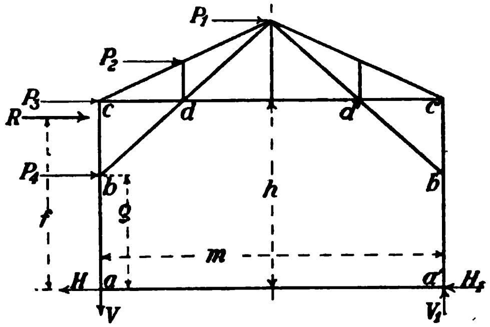


Fig. 364.

Bending Stress in the Columns Supporting a Roof Truss shown in Fig. 364.—Fig. 364 shows a roof truss supported on columns, and subjected to horizontal forces P_1, P_2, P_3 and P_4 due to wind or to any other cause.

Two cases arise, one when the columns are fixed at their bases *a* and *a'*, the other when they are hinged at these points.

Columns hinged at *aa'*. The horizontal forces may be replaced by their resultant R acting with a leverage *f*. It will be sufficiently exact to consider the horizontal reactions at *a* and *a'* as each equal to $\frac{R}{2}$—

$$H = H_1 = \frac{R}{2}.$$

Take moments about *a'*—

$$Rf - Vm = 0 \text{ or } V = \frac{Rf}{m}.$$

Figs. 365 and 366 represent the forces acting on the posts *ac* and *a'c'* respectively; Fig. 365 shows the panel loads acting on the truss.

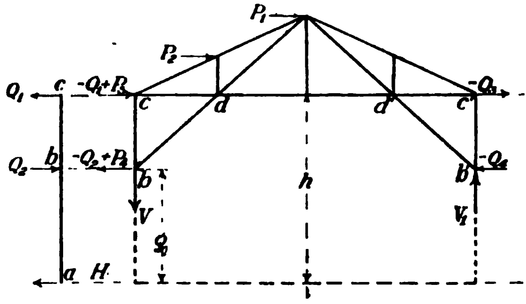


FIG. 365.

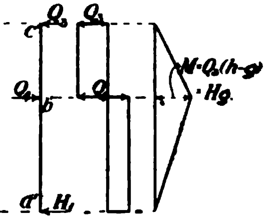


FIG. 366.

The forces causing bending in *ac* and *a'c'* are denoted by Q_1, Q_2, Q_3 and Q_4; they are equal and opposite to P_1, P_2, P_3 and P_4 respectively. Take moments about *b*, then—

$$Hg = Q_1(h - g) \text{ and } Q_1 = \frac{Hg}{h - g}.$$

$$Q_2 = H + Q_1.$$

Fig. 366 shows the diagrams of shearing stresses and bending moments in the post *ac*.

The forces P and reactions Q must be combined where they act at the same panel points in the following manner:—

At *c* the load is $- Q_1 + P_3$
,, *c'* ,, $- Q_3$
,, *b* ,, $- Q_2 + P_4$ also V
,, *b'* ,, $- Q_4$ also $- V_1$.

The stresses in the framework (Fig. 364) may be found for these loads by means of a reciprocal diagram, and the direct stresses combined with those due to bending.

Columns Fixed at a and a'.—The columns may be regarded as fixed at their bases a and a' and supported at c and c', hence it follows that there must be a point of inflection i somewhere between a and b. It can be shown that the distance ai has the value—[1]

$$ai = \frac{g}{2}\left(\frac{2h + g}{h + 2g}\right).$$

It is, however, sufficiently accurate for the ordinary proportions of knee-braced columns supporting roofs to assume that $ai = ib$.

A similar assumption is made in the case of the portal bracing of a bridge such as the Pratt truss, Plate II., where the point of inflection is taken midway between the unsupported lengths of the inclined posts. We also assume that $H = H_1 = \frac{R}{2}$. Take moments about i, then—

$$Q_1(h - ai) - Q_2(g - ai) = 0.$$

$$\text{Also } Q_1 = Q_2 - H.$$

$$\therefore Q_2 = \frac{H(h - ai)}{h - g}.$$

The force H at a is equal to a force H_2 at i and a moment H_2ai about the base, and for the inside $H_3 = H_1$ acting at i'.

Take moments about a', then—

$$Rf - H_2ai - H_3ai - Vm = 0.$$

But since $H_2 + H_3 = R$—

$$R(f - ai) = Vm = - V_1m.$$

The forces acting at the panel points, and the diagram of bending moments and shearing stresses, are shown in Figs. 367 and 368.

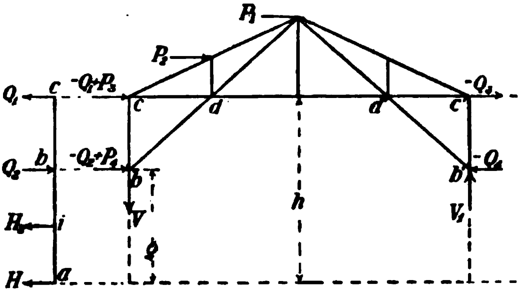


FIG. 367.

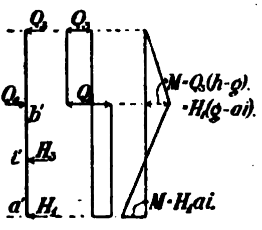


FIG. 368.

The direct stresses obtained in the various members of the framework for the loads shown in Fig. 367 must be determined, and the bending stresses shown in Fig. 368 must be combined with the direct stresses in the posts ac and $a'c'$ as before.

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[1] *Influence Lines*, by Messrs. Burr and Falk.

CHAPTER XV.

STRESSES IN BRACED PIERS.

BRACED piers must be designed not only to carry the dead and live load discharged by the main girders or trusses, but to resist the maximum horizontal force due to wind.

The stresses due to wind are considerable in lofty viaducts of this class. One of the best examples occurs on the New York, Lake Erie, and Western Coal and Railway Company's railway across the Kinzua Valley. The superstructure consists of lattice deck girders designed for a single line, spaced 10 ft. apart transversely, and alternately of 61 and 38½ ft. spans.

The towers or piers vary in height from 20 to 280 ft.; the total height from rail level to the water level in the Kinzua Creek is 301 ft. The piers are composed of Phœnix columns for the main compression members and transverse struts, with longitudinal struts of four latticed angles and diagonal tension rods arranged in pairs. Very high winds sweep through the gorge, which, taken in conjunction with the vibrations caused by passing trains on such a lofty viaduct, necessitate careful consideration.

The width of the piers at the top is 10 ft., and at the widest part of the base 103 ft.

The method of obtaining the stress in braced piers will be illustrated in the following example :—

Figs. 369 and 370 show two views of a braced pier of similar proportions to those adopted for the Kinzua viaduct. The wind is supposed to act on the right side, and the stresses on one of the frames of the pier will be investigated.

Let P_1 = the pressure on the train taken over one-half the spans r and s.

P_2 = the pressure on the main girders taken over the same distance as P_1.

w_1, w_2, w_3, and w_4 = the panel pressures acting at the four corners of the pier at each story.

H = the total horizontal force acting at the top of the pier.

W = the dead load of half the spans r and s, in addition to the live load distributed over them, considered as acting at the points u, v, Fig. 369.

W_1, W_2, W_3, and W_4 = the panel weights of the pier itself acting at the joints as shown in Fig. 369.

The remaining quantities are sufficiently explained in Fig. 369.

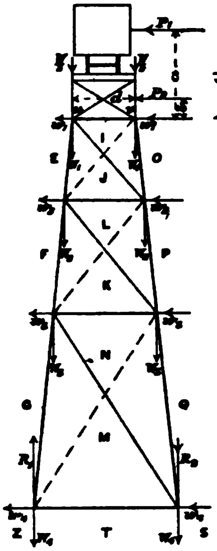

Fig. 369.

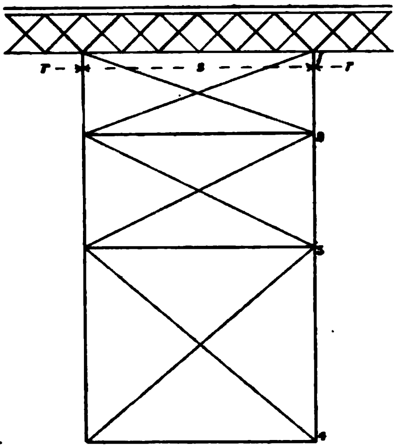

Fig. 370.

The wind pressures P_1 and P_2 on the train and main girders will relieve the pressure at v, and transfer an equal amount to u.

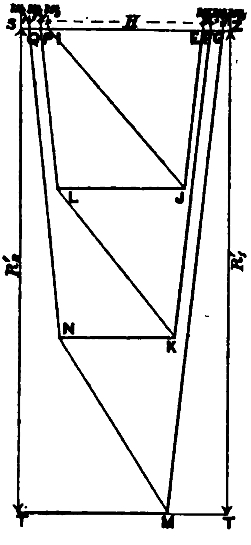

Fig. 371.

Let x denote this amount; then—

$$x = \frac{P_1 a + P_2 b}{d}.$$

Therefore the resultant reactions at u and v will be respectively—

$$\frac{W}{2} + x, \text{ and } \frac{W}{2} - x.$$

The horizontal component of x must be added to the horizontal forces P_1 and P_2, and to the panel pressures w_1, to find the total force acting at the top of the pier—

$$H = P_1 + P_2 + x \tan\theta + 2w_1.$$

We can set off H and the panel pressures w_2, w_3, and w_4 on a horizontal line as a force polygon, and obtain the reciprocal polygon in the usual way, thus determining the wind stresses, Fig. 371.

The post OI and the diagonals indicated by the dotted lines are unstressed when the direction of the wind is as indicated. If the wind acts in the opposite direction, the post EJ will be unstressed as well as the bars IJ, LK, and MN. The force Y acting at the base of the pier is represented in magnitude by the length of the line TT in the reciprocal polygon, Fig. 371. It tends to produce sliding at the bed-plates, and must be resisted by the friction developed by the total reactions due to the live and dead load added to the reactions measured from Fig. 371, or by the horizontal resistance at the feet of the posts due to these reactions added to the shearing resistance of the bolts.

The stresses obtained by measuring the lines in Fig. 371 must be added to those produced by the live and dead loads, thus—

TABLE LXVIII.

Member of structure, Fig. 369.	Stress due to wind pressure obtained from Fig. 371.	Stresses due to live and dead loads.
JE	- JE	$-\left(\frac{W}{2}+x+W_1\right)\sec\theta$
KF	- KF	$-\left(\frac{W}{2}+x+W_1+W_2\right)\sec\theta$
MG	- MG	$-\left(\frac{W}{2}+x+W_1+W_2+W_3\right)\sec\theta$
IO	—	$-\left(\frac{W}{2}-x+W_1\right)\sec\theta$
LP	+ LP	$-\left(\frac{W}{2}-x+W_1+W_2\right)\sec\theta$
NQ	+ NQ	$-\left(\frac{W}{2}-x+W_1+W_2+W_3\right)\sec\theta$
IE	- IE	$-\left(\frac{W}{2}+W_1\right)\tan\theta$
JL	- JL	$-W_2\tan\theta$
KN	- KN	$-W_3\tan\theta$
TM	- TM	$+\left(\frac{W}{2}-x+W_1+W_2+W_3\right)\tan\theta$
IJ	+ IJ	
LK	+ LK	
NM	+ NM	

The total reactions R_1 and R_2 may also be found by adding the vertical reactions due to wind pressure as measured from Fig. 371 to the total live and dead load on each side of the frame, thus—

$$R_1 = \frac{W}{2} - x + W_1 + W_2 + W_3 + W_4 + R_1' \text{ (Fig. 371).}$$

$$R_2 = \frac{W}{2} + x + W_1 + W_2 + W_3 + W_4 + R_2' \text{ (Fig. 371).}$$

It will generally be necessary to investigate the stresses in the pier when the live load is not on the main girders, as it may happen, for instance, that the tension in the diagonal members will be increased; 50 lbs. per sq. ft. of exposed area, estimated as in the case of trusses, should be allowed when the train is off the structure, and 30 lbs. when the train is on the structure. The maxima stresses due to either case should be used in designing the structure. The same remarks as to initial tension and intensities of working stress in the lateral systems of main girders and trusses apply without modification to braced piers.

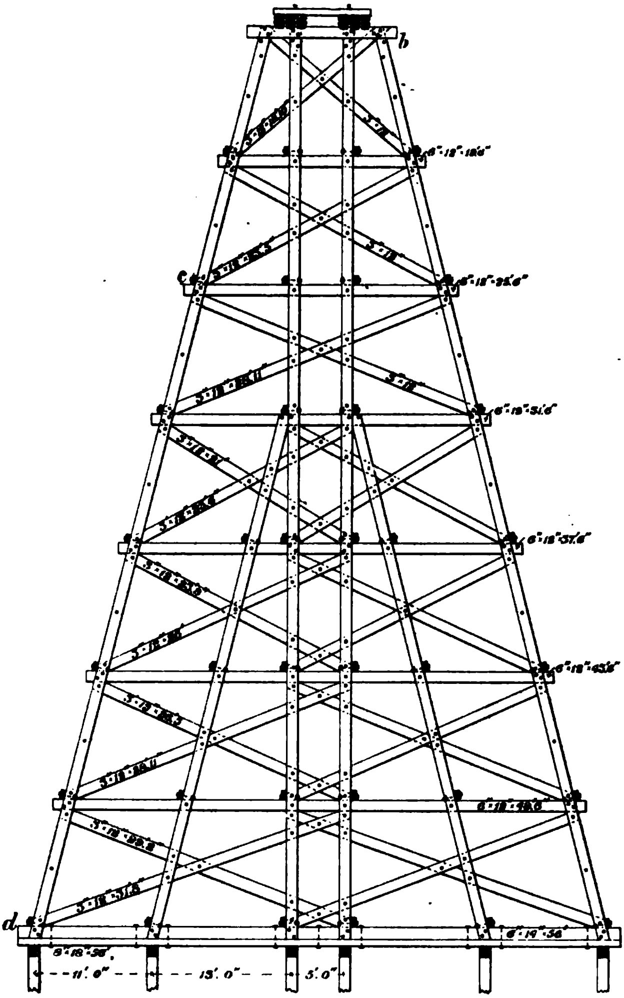

FIG. 372.

In a braced pier carrying a double line of railway, the maxima stresses in the diagonal members of the pier will occur when the live load is on the same side as the wind, the lee side line of rails being empty.

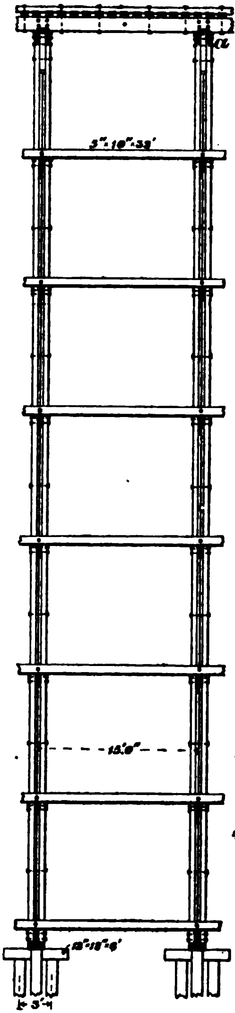

FIG. 373.

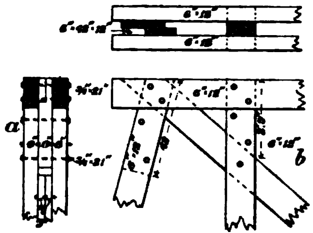

FIG. 374.

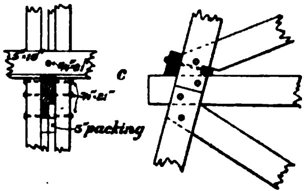

FIG. 375.

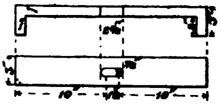

FIG. 376.

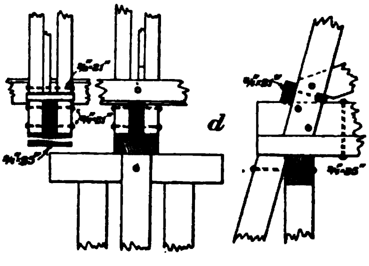

FIG. 377.

The stresses due to the horizontal traction of the train should be added to those already found. If the line is curved, the force acting along the radius of the curve due to the motion of the train in the curve should also be considered.

The stresses in Eiffel towers may be investigated in a similar manner to that indicated for a braced pier, the only vertical forces being the weight of the structure and the loads on the various floors which may be considered as acting vertically at the panel-points. The stresses due to these loads should be added to those obtained by drawing a reciprocal polygon for the horizontal forces due to wind. The inclination of NQ and MG may easily be adjusted from the diagram of wind pressure, so that the diagonal members MN are not necessary.

Timber Trestle Piers.—Figs. 157 and 165 illustrate the method of constructing trestle piers or bents of timber in viaducts of moderate height. When the height of the trestle pier is under 10 ft. it is not usual to provide sway bracing, and from 10 to 20 ft. sway braces are arranged as shown in Fig. 157. The braces should be bolted to the cap, to each pile or post, and to the sill with bolts ¾ in. in diameter, with cast-iron washers under both the head and the nut.

Where the height of the pier exceeds 20 ft., the upper panel should be made from 15 to 20 ft. and the odd lengths put in the lower panel.

Figs. 372 and 373 illustrate a high timber trestle pier used on the Californian and Central Railway, United States, America. These high piers require the greatest care and attention on the part of the designer. High trestle piers of timber present great varieties in design. They should be thoroughly sway-braced, each story having one set of braces. Counter-posts are also used to stiffen the bent, and longitudinal bracing between the bents, either in every bay or in every third or fourth bay, arranged horizontally and diagonally. A good plan is to brace every third bay diagonally so as to form a tower, and to provide simply horizontal longitudinal braces between the bents in the intermediate bays at each story.

CHAPTER XVI.

APPROXIMATE WEIGHTS OF STRINGERS. CROSS-GIRDERS, PLATE WEB GIRDERS AND TRUSSES. LIVE LOADS UPON BRIDGES. CONCENTRATED WHEEL LOAD DIAGRAM.

THE weights of portions of a structure, or the whole structure, may easily be found when the design is completed; but in order to determine the stresses and design of the structure it is necessary to know approximately the weight of the structure itself as part of the dead load to be carried. In the case of stringers, or floor beams of railway and highway bridges, it is not difficult to assume probable loads for the purposes of calculation deduced from the known weights of similar stringers and floor beams; or a tentative design may be roughly made in order to determine the weight. After the design is completed the actual weights may be compared with those assumed in the calculation, and the stresses recalculated if necessary. In ordinary bridges the dead load due to the weight of the structure itself is relatively small compared with the total load to be carried, and any small error in the assumed weights will not generally necessitate a recalculation of the stresses. In the case of stringers and floor beams we may use a formula—such as $w = al$, where w denotes the weight in pounds per lineal foot, l the span in feet, and a the constant depending on the load to be carried and the working stress adopted. In the case of a stringer of 25 ft. span to carry the wheel loads of Cooper's E_{50} locomotive, a is 7·2, $w = 25 \times 7{\cdot}2$, and the total weight of the stringer is $W = wl = 25^2 \times 7{\cdot}2 = 4500$ pounds, including the weight of half the lateral bracing attached to the top flanges. The weights per lineal foot of existing girders may be conveniently tabulated, or diagrams may be prepared from which the weight of a proposed girder may be approximately determined. Formulæ attempting to express the dead weight of a structure in terms of the span, loads and working stresses are not very satisfactory unless they are restricted to one particular type of structure, and in the case of main girders and trusses a formula—such as $w = al$—appears to be as good as any other, provided the constant a is well established from existing structures of a similar type.

Economic Depth in Plate Web Girders.—The most economical depth in a plate web girder is that which will produce flanges and web of approximately equal areas. Let l = span, d = depth, t = web thickness assumed uniform throughout the length, w = the load per lineal inch, M = the maximum moment in inch pounds, σ = the working stress, and v = the weight of a cubic inch of steel; then the volume of the web = ldt and the weight $vldt$, the stress in each flange = $\frac{M}{d}$, the area = $\frac{M}{\sigma d}$, and the weight = $\frac{Mlv}{\sigma t}$; or for two flanges the weight = $\frac{2Mlv}{\sigma t}$. Assuming the flanges to be of uniform cross-section, as is generally the case in cross-girders and stringers, the total weight is therefore—

$$W = vl\left(\frac{2M}{\sigma d} + dt\right).$$

This is a minimum when $\left(\frac{2M}{\sigma d} + dt\right)$ is a minimum, or when $\frac{dW}{d(d)} = 0$. $\therefore\ -\frac{2M}{\sigma d^2} + t = 0$, and $d^2 = \frac{2M}{\sigma t}$; since $M = \frac{wl^2}{8}$, where w is the equivalent distributed load, $d^2 = \frac{wl^2}{4\sigma t}$, also $\frac{2M}{\sigma d} = dt$, and $vl\frac{2M}{\sigma d} = vldt$, or the weight of the flanges equals that of the web.

To find the economical depth of a stringer 25 ft. span to carry locomotives equivalent to those of Cooper's E_{40}, it is seen from Table LXX. that the maximum moment on one stringer is 308,250 foot pounds, or 3,699,000 inch pounds, the value of w = 3945 pounds per ft., or 329 pounds per in., therefore—

$$d^2 = \frac{wl^2}{4\sigma t}.$$

Let σ = 10,000 pounds per sq. in., and $t = \frac{1}{2}$ in., then—

$$d^2 = \frac{329 \times 25^2 \times 12^2}{4 \times 10000 \times \frac{1}{2}} = 1480 \therefore d = 38{\cdot}4 \text{ in.}$$

We may write—

$$d = \frac{l}{2}\sqrt{\frac{w}{\sigma t}}, \text{ or } d = \sqrt{\frac{2M}{\sigma t}}.$$

It should be observed that variations in the value of d will not cause a corresponding variation in the weight.

Dead Loads upon Railway Bridges.—In a single line through railway bridge of from 150 to 350 ft. span, pin connected with open American deck weighing about 400 pounds per lineal foot, Messrs. Waddell and Hendrick have proposed the following formula for the purposes of calculation—

$$w = 8{\cdot}63(l + 1{\cdot}3W - 140),$$

where w is the dead load on the two trusses in pounds per lineal foot, W_1 the weight of one of the two locomotives assumed at the head of the train in tons of 2000 pounds, and l the length of the span in feet. For a double line through bridge the value of the dead load found by the above formula should be increased by 85 per cent. If $l = 200$, and $W = 177{\cdot}5$ corresponding with Cooper's E_{50} type, then—

$$w = 8{\cdot}63(200 + 1{\cdot}3 \times 177{\cdot}5 - 140) = 2509$$

or 1254·5 pounds per foot run for each truss. If there are eight panels, each 25 ft. long, the dead load at each panel point is—

$$25 \times 1254{\cdot}5 = 31362{\cdot}5 \text{ pounds.}$$

This value may be checked in the following manner:—

The weight of the two trusses per lineal foot is—

$$w = al = 7{\cdot}6 \times 200 = 1520 \text{ pounds.}$$

The weight of the deck is 400 pounds per lineal foot, and of the floor beams 520 pounds per lineal foot. The panel point dead load is—

$$\text{Track} = \frac{400}{2} \times 25 = 5{,}000$$

$$\text{Floor} = \frac{520}{2} \times 25 = 6{,}500$$

$$\text{Truss} = \frac{1520}{2} \times 25 = 19{,}000$$

Total dead load at panel point = 30,500 pounds.

Again the total dead load for a single line bridge with open deck including track and bracing for Cooper's E_{50} varies from—

$$w = 1100 + 7l \text{ to } w = 1400 + 10l.$$

The variation is due to the design of the various details and the working stresses adopted.

For plate web girder bridges carrying a single track and Cooper's E_{40} the following formulæ may be used—

For deck bridges, $w = 550 + 9l$.
For through bridges, $w = 800 + 11l$.

For Cooper's E_{50}—

For deck bridges, $w = 690 + 11{\cdot}5l$.
For through bridges, $w = 1000 + 14l$.

If a ballasted road is laid over a steel deck, the dead load will be considerably increased and also the weight of the main girders or trusses, but formulæ could easily be arranged to give the dead load for any type of deck.

Loads upon Highway Bridges.—Messrs. Waddell and Hendrick have proposed the following formula for highway bridges carrying electric cars with 24,000 pounds on two axles spaced 7 ft. centres, or a steam roller on two axles spaced 11 ft. centres; the front roller is 4 ft. wide, and each of the two rear wheels is 20 in. wide spaced laterally 5 ft. centres. The axle loads are 12,000 and 18,000 pounds on the front and rear respectively—

$$w = 34 + 226 + 0{\cdot}16bl + 0{\cdot}7l,$$

where b denotes the width of the roadway and footpaths, and l the span.

If $l = 150$ ft., and $b = 20$ ft., $w = 845$ pounds per lineal foot.

In highway bridges for city and suburban traffic, the loads vary considerably, more especially when they are traversed by one or more lines of electric cars. The requirements of these bridges at the present time suggest their division into certain classes. The specifications of the American Bridge Company in regard to highway bridges is as follows:—

1. Bridges under these specifications are divided into six classes, *viz.*—

Class A.—For city traffic.
Class B.—For suburban or interurban traffic with heavy electric cars.
Class C.—For country roads with light electric cars or heavy highway traffic.
Class D.—For country roads with ordinary highway traffic.
Class E_1.—For heavy electric street railways only.
Class E_2.—For light electric street railways only.

2. All structures to be of rolled steel, except the flooring and wheel-guards of Classes A, B, C, E_1 and E_2, and the stringers, flooring and wheel-guards of Class D, which may be of timber. Cast iron or cast steel will be permitted only in machinery of movable bridges and in special cases for shoes and bearings.

3. The following types of bridges are recommended:—

For spans up to 25 ft.—Rolled beams.
For spans from 25 to 40 ft.—Rolled beams or plate girders.
For spans from 40 to 80 ft.—Plate or lattice girders.
For spans from 80 to 140 ft.—Lattice girders.
For spans over 140 ft.—Lattice girders or pin-connected trusses.

4. At an elevation of 1 ft. and over above the roadway, the clear distance between trusses shall be at least 14 in. greater than the width of the roadway between wheel-guards.

5. For Classes A, B, C, E_1 and E_2 the clear headroom for a width of 6 ft. over each track shall not be less than 15 ft., for Class D not less than 12½ ft. above the floor.

6. For bridges carrying electric cars the clear width from the

centre of track shall not be less than 6½ ft. at a height exceeding 1 ft. 6 in. above the top of rails, where the tracks are straight.

7. In determining the clearance on curves the extreme length of electric car shall be taken as 45 ft., the width 8 ft., and the distance between centres of trucks as 20 ft., unless otherwise specified.

8. The width between centres of trusses shall in no case be less than one-twentieth of the span between centres of end pins or shoes.

9. A strong and suitable handrailing shall be placed at each side of the bridge, except where plate girders serve the same purpose, and be rigidly attached to the superstructure.

10. Each bridge shall have such name plates suitably inscribed and located as may be required.

11. All floor beams in through bridges shall be riveted to the main girders.

12. Steel stringers shall preferably be riveted to the web of the floor beams.

Wooden joists shall not be less than 3 in. thick, shall be spaced not more than 2½ ft. between centres, and shall be lapped over the seat angles or floor beams to exact level. In the latter case they shall lap by each other over the full width of the floor beam, and shall be separated ½ in. for free circulation of air.

13. For single thickness the roadway planks shall not be less than 3 in. thick, nor less than one-twelfth of the distance between stringers, and shall be laid transversely with ½ in. openings.

14. When an additional wearing surface is specified for the roadway, it shall be 1½ in. thick, and the lower planks, of a minimum thickness of 2½ in., shall be laid diagonally and with ½ in. openings.

15. Wheel-guards of a cross-section not less than 6 in. by 4 in. on each side of the roadway shall be provided. They shall be blocked up from the floor plank with blocks 2 in. by 6 in. by 12 in. long, not over 5 ft. apart centre to centre, held in place by one ¾ in. bolt passing through the centre of each blocking piece and securely fastened to the stringer below. The wheel-guards shall be spliced with half and half joints with 6 in. lap over a blocking piece.

16 The footwalk planks shall not be less than 2 in. thick nor more than 6 in. wide, spaced with ½ in. openings.

17. All planks shall be laid with the heart side down; shall have full and even bearing on and be firmly attached to the stringers.

18. For bridges of Classes A and B a solid floor, consisting of stone, asphalt, etc., on a concrete bed, is recommended. For this case the flooring will consist of buckle-plates or corrugated sections, and the concrete bed shall be at least 3 in. thick for the roadway, and 2 in. thick for the footwalk, over the highest point to be covered, not counting rivet or boltheads.

19. Buckleplates shall not be less than $\frac{5}{16}$ in. thick for the roadway and $\frac{1}{4}$ in. thick for the footwalk.

20. For solid floor the curb holding the paving and acting as a wheel-guard on each side of the roadway shall be of stone or steel projecting about 6 in. above the finished paving at the gutter. The curb shall be so arranged that it can be removed and replaced when worn or injured. There shall also be a metal edging strip on each side of the footwalks to protect and hold the paving in place.

21. Provision shall be made for drainage clear of all parts of the metal work.

22. The floor of bridges of Classes E_1 and E_2 shall consist of cross-ties not less than 6 in. by 6 in., spaced with openings not exceeding 6 in. and securely fastened to the stringers by bolts. There shall be guard timbers not less than 6 in. by 6 in. on each side of each track, with their inner faces not less than 9 in. from centre of rail. They shall be notched 1 in. over every tie, and fastened to every fourth tie.

LOADS.

23. In determining the weight of the structure for the purpose of calculating stresses, the weight of timber shall be assumed at 4 pounds per ft., B. M., the weight of concrete and asphaltum at 130 pounds, of paving brick at 150 pounds, and of granite stone at 160 pounds per cub. ft.

The rails, fastenings, splices and guard timbers of street railway tracks, resting on cross ties, shall be assumed as weighing 100 pounds per lineal ft. of track.

24. The bridges of the different classes shall be designed to carry, in addition to their own weight and that of the floor, a moving load, either uniform or concentrated, or both, as specified below, placed so as to give the greatest stress in each part of the structure.

CLASS A.—*City Bridges.*

For the floor and its supports, on each street car track or on any part of the roadway, a concentrated load of 24 tons on two axles 10 ft. centres and 5 ft. gauge (assumed to occupy a width of 12 ft.), and upon the remaining portion of the floor, including foot walks, a load of 100 pounds per sq. ft.

For the trusses, for spans up to 100 ft., 1800 pounds per lineal ft. of each car track (assumed to occupy 12 ft. in width), and 100 pounds per sq. ft. for the remaining floor surface; for spans of 200 ft. and over, 1200 pounds for each lineal ft. of track and 80 pounds per sq. ft. of floor; proportionally for intermediate spans.

Class B.—*Suburban or Interurban Bridges.*

For the floor and its supports, on any part of the roadway, a concentrated load of 12 tons on two axles, 10 ft. centres and 5 ft. gauge (assumed to occupy a width of 12 ft.), or on each street car track a concentrated load of 24 tons on two axles 10 ft. centres, and upon the remaining portion of the floor, including footwalks, a load of 100 pounds per sq. ft.

For the trusses, for spans up to 100 ft., 1800 pounds per lineal ft. of each car track and 80 per sq. ft. for the remaining floor surface; for spans of 200 ft. and over, 1200 pounds for each lineal ft. of track and 60 pounds per sq. ft. of floor; proportionally for intermediate spans.

Class C.—*Heavy Country Highway Bridges.*

For the floor and its supports, on any part of the roadway, a concentrated load of 12 tons on two axles 10 ft. centres and 5 ft. gauges (assumed to occupy a width of 12 ft.), or on each street car track a concentrated load of 18 tons on two axles 10 ft. centres; and upon the remaining portion of the floor, including footwalks, a load of 100 pounds per sq. ft.

For the trusses, same as for Class B, except load on car tracks for spans up to 100 ft. will be 1200 pounds, and for spans of 200 ft. and over, 1000 pounds.

Class D.—*Ordinary Country Highway Bridges.*

For the floor and its supports, a load of 80 pounds per sq. ft. of total floor surface, or 6 tons on two axles 10 ft. centres and 5 ft. gauge.

For the trusses, a load of 80 pounds per sq. ft. of total floor surface for spans up to 75 ft.; and 55 pounds for spans of 200 ft. and over; proportionally for intermediate spans.

Class E_1.—*Bridges for Heavy Electric Street Railways only.*

For the floor and its supports, on each track a load of 24 tons on two axles 10 ft. centres.

For the trusses, a load of 1800 pounds per lineal ft. of each car track for spans up to 100 ft.; and a load of 1200 pounds for spans of 200 ft. and over; proportionally for intermediate spans.

Class E_2.—*Bridges for Light Electric Street Railways only.*

For the floor and its supports, on each track a load of 18 tons on two axles 10 ft. centres.

For the trusses, a load of 1200 pounds per lineal ft. of each car track for spans up to 100 ft.; and a load of 1000 pounds for spans of 200 ft. and over; proportionally for intermediate spans.

25. To compensate for the effect of impact and vibration, 25 per cent. of the maximum stresses resulting from the above-mentioned live load shall be added thereto.

26. The wind pressure shall be assumed acting in either direction horizontally:—

First. At 30 pounds per sq. ft. on the exposed surface of all trusses and the floor as seen in elevation, in addition to a horizontal live load of 150 pounds per lineal ft. of the span moving across the bridge.

Second. At 50 pounds per sq. ft. on the exposed surface of all trusses and the floor system. The greatest result shall be assumed in proportioning the parts.

27. For longitudinal bracing of structures carrying street railroads, the momentum produced by suddenly stopping the train shall be considered; the coefficient of friction of wheels sliding upon the rails being assumed as 0·2.

28. When the structure carrying a street railroad is on a curve, the additional effects due to the centrifugal force shall be considered.

TABLE LXIX.

UNIFORM LIVE LOADS FOR THE TRUSSES.

CLASS A.			CLASS B.		
Span in ft.	Pounds per lineal ft. of each street car track.	Pounds per sq. ft. of remaining floor surface.	Span in ft.	Pounds per lineal ft. of each street car track.	Pounds per sq. ft. of remaining floor surface.
UP TO 100	1800	100	UP TO 100	1800	80
105	1770	99	105	1770	79
110	1740	98	110	1740	78
115	1710	97	115	1710	77
120	1680	96	120	1680	76
125	1650	95	125	1650	75
130	1620	94	130	1620	74
135	1590	93	135	1590	73
140	1560	92	140	1560	72
145	1530	91	145	1530	71
150	1500	90	150	1500	70
155	1470	89	155	1470	69
160	1440	88	160	1440	68
165	1410	87	165	1410	67
170	1380	86	170	1380	66
175	1350	85	175	1350	65
180	1320	84	180	1320	64
185	1290	83	185	1290	63
190	1260	82	190	1260	62
195	1230	81	195	1230	61
200 AND OVER.	1200	80	200 AND OVER.	1200	60

TABLE LXIX.—*continued.*

CLASS C.

UP TO					
100	1200	80	155	1090	69
105	1190	79	160	1080	68
110	1180	78	165	1070	67
115	1170	77	170	1060	66
120	1160	76	175	1050	65
125	1150	75	180	1040	64
130	1140	74	185	1030	63
135	1130	73	190	1020	62
140	1120	72	195	1010	61
145	1110	71	200	1000	60
150	1100	70	AND OVER.		

CLASS D.		CLASS E_1.		CLASS E_2.	
Span in ft.	Pounds per sq. ft. of floor surface.	Span in ft.	Pounds per lineal ft. of each car track.	Span in ft.	Pounds per lineal ft. of each car track.
UP TO		UP TO		UP TO	
75	80	100	1800	100	1200
80	79	105	1770	105	1190
85	78	110	1740	110	1180
90	77	115	1710	115	1170
95	76	120	1680	120	1160
100	75	125	1650	125	1150
105	74	130	1620	130	1140
110	73	135	1590	135	1130
115	72	140	1560	140	1120
120	71	145	1530	145	1110
125	70	150	1500	150	1100
130	69	155	1470	155	1090
135	68	160	1440	160	1080
140	67	165	1410	165	1070
145	66	170	1380	170	1060
150	65	175	1350	175	1050
155	64	180	1320	180	1040
160	63	185	1290	185	1030
165	62	190	1260	190	1020
170	61	195	1230	195	1010
175	60	200	1200	200	1000
180	59	AND OVER.		AND OVER.	
185	58				
190	57				
195	56				
200	55				
AND OVER.					

LIVE LOADS UPON RAILWAY BRIDGES.

In railway bridges it is the common practice in Great Britain to estimate the live load in tons per foot run, which would produce the same stresses as those occurring when the heaviest engines and trains in use, or contemplated, are passing over the structure.

To some extent this practice obtains in America, and it has been strongly advocated by Dr. Waddell.[1] Such loads will vary with the span length, and also as they are used for the determination of the moments or shearing stresses. The following table shows the uniform load per lineal foot which will produce the same moments and shears in a given span as the actual concentrated loads of Cooper's Class E_{40} standard loading shown in Fig. 378 :—

TABLE LXX.

MAXIMUM MOMENTS, M, END SHEARS, S, AND FLOOR-BEAM REACTIONS, R, PER TRACK, PRODUCED BY COOPER'S STANDARD LOADING E_{40}, ON SPANS FROM 10 TO 250 FT.

Span in ft.	Max. moment M, in foot lbs.	Max. end shear, S, in lbs.	Max. floor-beam reaction, R in lbs.	Equivalent uniform load.		
				M.	S.	R.
10	112,500	60,000	82,000	9000	12,000	8200
12	169,100	68,000	93,200	9340	11,330	7830
15	254,000	80,000	110,000	9030	10,670	7330
18	341,600	92,000	122,000	8430	10,240	6780
20	400,000	100,000	130,000	8000	10,000	6500
25	616,500	113,500	151,000	7890	9,080	6040
30	820,800	126,500	171,500	7300	8,430	5720
35	1,052,500	138,000	195,000	6870	7,890	5570
40	1,312,500	151,500	216,000	6560	7,570	5400
50	1,908,300	176,400	257,000	6110	7,060	5140
60	2,580,000	196,000	305,000	5730	6,530	5080
70	3,372,000	222,000	354,000	5510	6,340	5060
80	4,320,000	250,000	396,000	5400	6,250	4950
90	5,320,000	276,000	437,000	5250	6,130	4860
100	6,436,000	301,200	474,000	5140	6,020	4740
125	9,960,000	360,900	—	5100	5,770	—
150	14,100,000	417,600	—	5010	5,570	—
175	18,720,000	467,800	—	4890	5,350	—
200	23,700,000	524,400	—	4740	5,240	—
250	35,220,000	629,300	—	4510	5,030	—

It will be observed that the equivalent uniform load for producing maximum shears is always greater than that required to produce the maximum moment, although the excess is less with long spans than short spans. The present practice in America requires bridges to be designed for the standard type of locomotives which are intended to cover any of the actual loads produced by the heaviest locomotives in use on the railway. The standard locomotives and train loads devised by Cooper are extensively used by railroad companies in America, and they possess considerable advantages. The length of wheel bases for all these locomotives is the same, and are expressed in feet without the odd inches. The weights on the axles all vary in the same proportion in any two classes. Hence if the stresses are found for one class of loading, the stresses in the same structure for a different class may be determined by multiplying by the correct ratio. Fig. 378 shows Cooper's standard loading, and it will be seen that E_{40} denotes 40,000 pounds on each of the four driving wheels, 20,000 on the leading wheels, and 26,000 on each of the four tender wheels. To find the corresponding loads on another class, such as E_{30}, it is only necessary to multiply by the ratio $\frac{3}{4}$.

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[1] *Transactions American Society of Civil Engineers*, vol. xxvi., February and March, 1892. Also in *The Principal Professional Papers*, 1905 : published by Virgil H. Lewis, New York City.

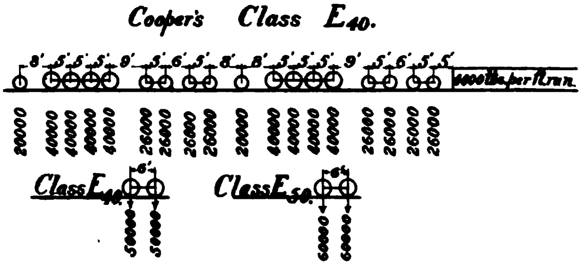


FIG. 378.

Dr. Waddell has proposed seven types of standard loading, denoted by T, U, V, W, X, Y and Z. Two wheel loads, 60 to 20 per cent. heavier than the load on the driving wheels, are taken to represent the engines with two coupled axles, and are to be used for those portions of the structure where they produce greater stresses than those produced by the two typical locomotives. Fig. 379 shows

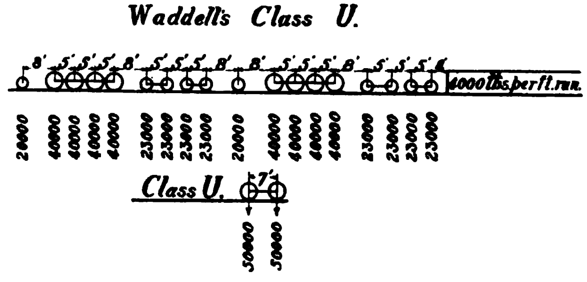


FIG. 379.

Waddell's Class U. The weight of one of the engines and tenders is 136 tons of 2000 pounds. Like Cooper's standards, the wheel spacing is the same in the seven classes, and the loads are proportional in a similar manner.

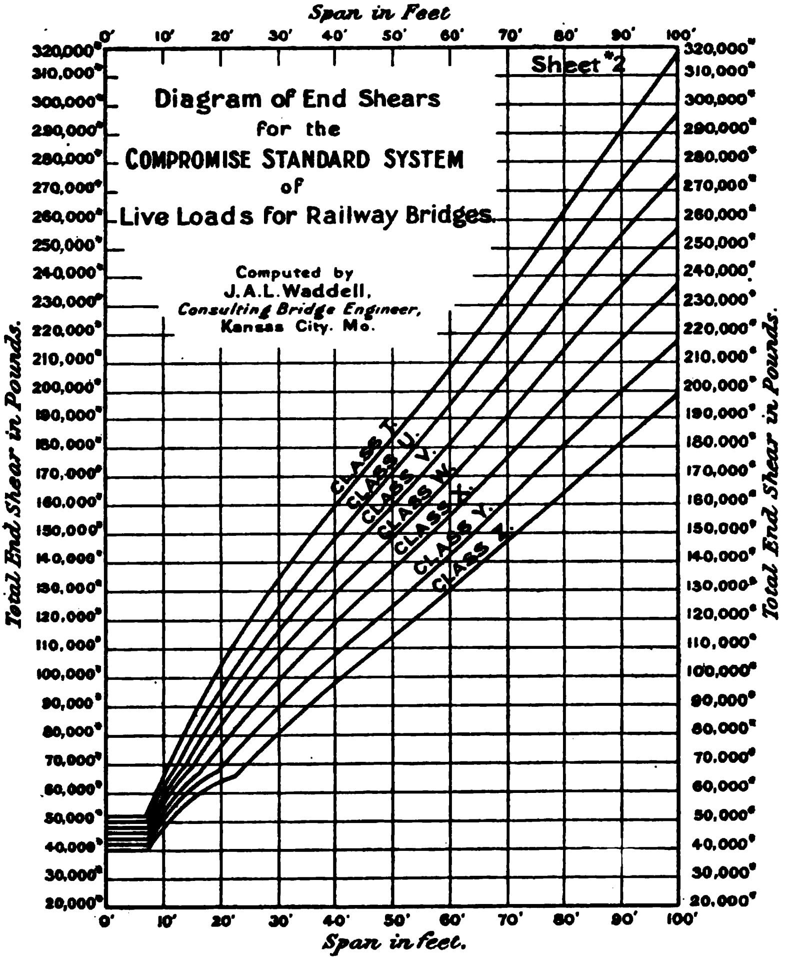
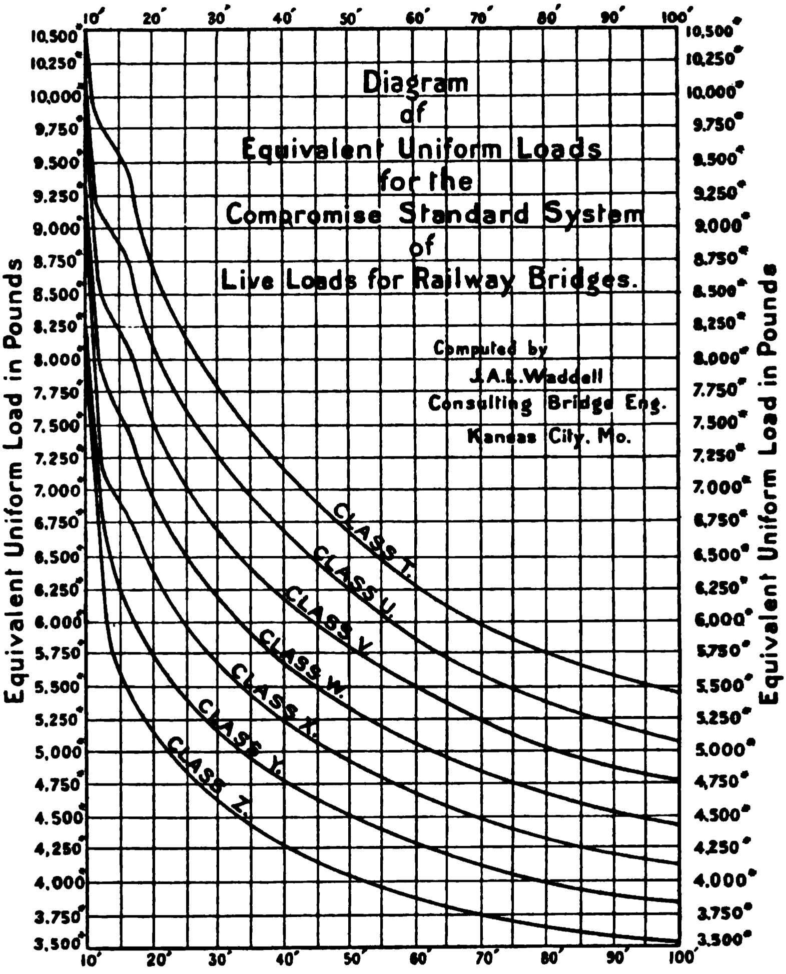


FIG. 380.

These seven classes cover the loadings on almost every kind of railway bridge, and the engineer may select the particular one suited to his requirements. Fig. 379 would be a reasonable type for bridges in India and New South Wales, the axle loads being about ten per cent. greater than those in actual use on these lines. Cooper's Class

E_{50} or Waddell's Class T are still heavier. Waddell's Class Z is the lightest, having 15, 25, and 18 units of 1000 pounds on the pilot, driving and tender wheels respectively. Class T, the heaviest, has 21, 43 and 24 units of 1000 pounds on the pilot, driving and tender wheels respectively. In Class Z the two axle loads of 40,000 pounds are spaced



Span in Feet.

N.B.—For panels or spans shorter than 10 ft. use one of the axle loads of the alternative loading (placing the wheel at mid length) and the formula $M = \frac{1}{4}$ WL.

FIG. 381.

seven feet apart, and in Class T they are 52,000 pounds—also spaced seven feet apart.

Dr. Waddell has prepared the diagrams, Figs. 380 to 382, showing

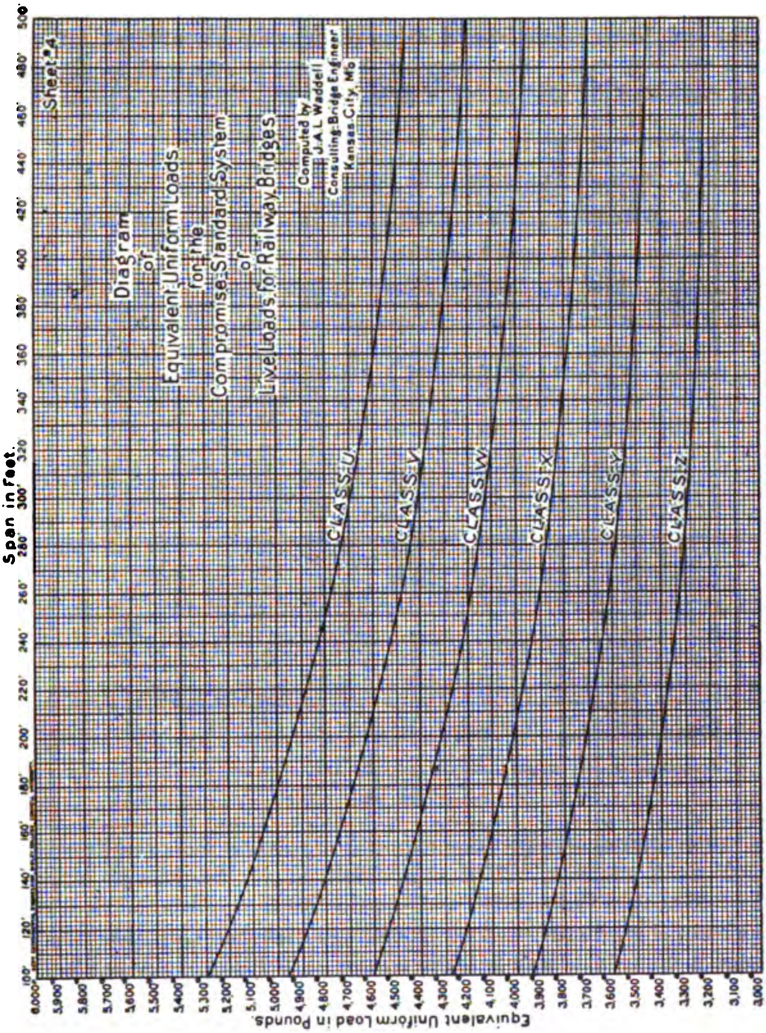


FIG. 382.

the equivalent uniformly distributed load in pounds producing practically the same moments and shears as the typical locomotives in the seven classes. The calculated stresses obtained by both methods showed that the equivalent load system produced results closely agreeing with those obtained by the concentrated load system.

Diagram for Concentrated Load System (Fig. 383).—Many different devices have been proposed by engineers for reducing the labour involved in calculating the stresses in the various members of a truss or girder when loaded with one or more locomotives followed by a uniform load. The diagram, Fig. 383, is intended to illustrate this method of calculation in a simple manner. The truss was supplied by the American Bridge Company and represents their standard for a single track bridge of 150 ft. span (Plate II.). The loading consists of two standard locomotives of Cooper's Class E_{50} followed by a train represented by a uniform load of 5000 pounds per foot run. In the diagram the unit of load is 1000 pounds, and the unit moment is 1000 foot pounds. The distance of each axle from the first is given in column 1. The half axle loads is given in column 2, and the sum of each load and all preceding loads on the left is given in column 3. The moment of all preceding loads with reference to the axle of each wheel is given in column 4, and the number of each axle is given in column 5. The loads and moments apply to one truss of a single track bridge, as well as the shears and stresses deduced from them.

5 Nº. of axle	4 Mom.t about each axis of all preceding loads	3 Sum of each load and all preceding loads	2 Half axle loads	1 Dist. of each axle from Nº. 1
1	0·00	12·50	12·50	0
2	100·00	37·50	25·00	8
3	287·50	62·50	25·00	13
4	600·00	87·50	25·00	18
5	1037·50	112·50	25·00	23
6	2050·00	128·75	16·25	32
7	2693·75	145·00	16·25	37
8	3563·75	161·25	16·25	43
9	4370·00	177·50	16·25	48
10	5790·00	190·00	12·50	56
11	7310·00	215·00	25·00	64
12	8385·00	240·00	25·00	69
13	9585·00	265·00	25·00	74
14	10910·00	290·00	25·00	79
15	13520·00	306·25	16·25	88
16	15051·25	322·50	16·25	93
17	16986·25	338·75	16·25	99
18	18684·00	355·00	16·25	104
	20455·00	355·00		109
	22630·00	370·00		115
	24511·00	382·50		120
	26455·00	395·00		125
	28461·25	407·50		130
	30530·00	420·00		135
	32681·23	432·50		140
	34855·00	445·00		145
	37111·25	457·50		150
	unit. 1000 ft. lbs.	unit. 1000 lbs.		unit. 1 ft.

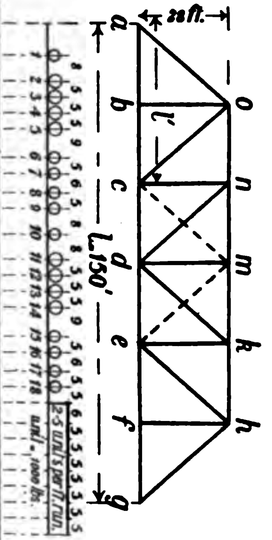


FIG. 383.

For instance, the distance of wheel 14, the last driving wheel of the second locomotive, to wheel 1 is 79 ft., column 1. The sum of the loads 1 to 14 is given in column 3 as 290 units, and the moment of all the loads to the left of wheel 14 about wheel 14 is given in column 4 as 10,910 units representing 10,910,000 foot pounds.

Each moment may be derived from the one preceding it by calculating the moment increment; thus for the moment about the wheel 15, the lever arm of each load is increased by 9 ft., hence the increment, 290 × 9 = 2610, added to 10,910, gives 13,520 units, column 4.

For any point at a distance x, measured from the head of the uniform load to the right, the general expression for the moment about this point is—

$$Wy + Wx + \frac{wx^2}{2}.$$

Where W = the resultant of all the concentrated wheel loads.

y = the distance of the centre of gravity of the sum of the loads, or resultant, denoted by W.

w = the weight of the uniform train load per lineal foot = 2·5 units.

The value of Wy is given by the figure immediately under the head of the train load in column 4, as 20,455 units, while W is given on the same vertical line in column 3 as 355 units.

Since the uniform load is 2·5 units per foot run, we have—

$$20{,}455 + 355x + 1{\cdot}25x^2 = \text{moment about } x.$$

If we put $x = 6, 11, 16$ feet respectively in the above expression, we obtain the value of the moments at a distance from the first axle corresponding with 115, 120 and 125 feet respectively as shown on the diagram Fig. 383; thus if $x = 16$ ft., we obtain—

$$20{,}455 + 355 \times 16 + 1{\cdot}25 \times 16 \times 16 = 26{,}455 \text{ units}.$$

A tracing of the truss diagram drawn to the same scale as the moment diagram should be prepared and placed directly over it. This tracing can be moved to the right or the left to represent the movement of the train upon the bridge as required. Let the wheel 11 be placed at the panel point of the truss indicated by d, then for this position the length of the uniform train load on the bridge is—

$$75 + 64 - 109 = 30 \text{ feet}.$$

Where 75, 64 and 109 are respectively the distances from g to d, from d to the wheel 1 and from wheel 1 to the head of the uniform load. The moments of all the loads about the right support g is—

$$20{,}455 + 355 \times 30 + 1{\cdot}25 \times 30^2 = 32{,}230.$$

The left reaction R is—

$$R = \frac{32230}{150} = 214 \cdot 86.$$

The moment at *d* is—

$$M_d = 214 \cdot 86 \times 75 - 7310 = 8805,$$

the moment of the loads on the left of *d* about *d* being taken directly from column 4 of the diagram, Fig. 383. When the panels are of equal length, the bending moment may be derived from the moment at the right support. In this case the lever arms are three and six panels respectively, and—

$$M_d = \tfrac{3}{6} \times 32{,}230 - 7310 = 8805, \text{ as before.}$$

Position of Wheel Loads for Maximum Moment and Shear. —To find the position of the axle loads on a truss to produce the maximum shear, and the maximum moment, we must use the criteria already found (Chap. IV.). The maximum shear at any point *d* will occur when one wheel of the train is on the panel point *d*, and the total load upon the bridge is *n* times the load in front of *d* (not including the one at *d*) as nearly as possible. Let P denote the loads in front of *d*, W the total load on the bridge, and *n* the number of panels in the truss, then we may write—

$$W - nP \text{ is a minimum.}$$

In the diagram, Fig. 383, this will occur when the wheel 3 is at *d*, as may be seen by moving the tracing so as to bring the wheel 4 and the wheel 5 also over the point *d*:—

Wheel 3 at *d*, $P = 37 \cdot 5$, $W = 290$,
$290 - 6 \times 37 \cdot 5 = 65$.
Wheel 4 at *d*, $P = 62 \cdot 5$, $W = 306 \cdot 25$,
$306 \cdot 25 - 6 \times 62 \cdot 5 = -68 \cdot 75$.
Wheel 5 at *d*, $P = 87 \cdot 5$, $W = 322 \cdot 5$,
$322 \cdot 5 - 6 \times 87 \cdot 5 = -202 \cdot 5$.

The shear in the panel *dc* is the reaction at *a*, less the loads between *a* and *d*, or: Let M_w denote the moment of all the loads about the right support *g*. Let M_P denote the moments of the loads to the left of the point *d* about *d*, then the shear in the panel *cd* may be expressed as follows:—

$$\frac{M_w - nM_P}{l} = \frac{13520 - 6 \times 287 \cdot 5}{150} = 78 \cdot 6 \text{ units} = 78{,}600 \text{ pounds.}$$

The criterion for the maximum moment may be expressed thus—

$$\frac{l'}{l} = \frac{P}{W}$$

where l' denotes the distance of *d* from the end of the span *a*, and P the load over the distance l'. If wheel 11 is at d, $P = 215$, and $W = 430$,

$$\frac{l'}{l} = \frac{215}{430} = \frac{1}{2}.$$

The criterion is exactly satisfied when the wheel 11 is at d, but it frequently happens that a wheel must be placed a little to the right, or to the left of the point about which the moment is required, in order to exactly satisfy the criterion. The moment about d for the position found is—

$$\frac{M_w l'}{l} - M_P = \frac{32230 \times 75}{150} - 7310 = 8805 \text{ units},$$

or 8,805,000 foot pounds.

The stress in the bar mn is found by dividing the maximum moment by the depth of the girder—

$$\frac{8805}{28} = 314{\cdot}464 \text{ units} = 314{,}464 \text{ pounds.}$$

CHAPTER XVII.

PLATE WEB GIRDER DECK BRIDGE.

THE following example will illustrate the method of designing a plate web girder in detail. The data is as follows :—

Total length of main girders = 70 ft.
Effective span „ „ = 68 „
Clear span „ „ = 66 „
Effective depth „ „ = 7 „

The material is structural steel, and the safe dead load working stress may be taken as 17,000 pounds per sq. in. in tension. All the live load stresses will be reduced to an equivalent dead load stress by adding an amount to the live load stress equal to the impact effect. All the rivets will be $\frac{7}{8}$ in. diameter, and the safe dead load stress in single and double shear may be taken as 6000 and 9000 pounds per rivet respectively. The safe bearing pressure may be taken as 20,000 pounds per sq. in., giving 7656 pounds as the value of a $\frac{7}{8}$ in. rivet in a $\frac{7}{16}$ in. plate.

The Dead Load consisting of the two main girders, lateral bracing, vertical frames, etc., the sleepers, or cross-ties, attached to the top of the girders, rails, etc., may be approximately estimated by the following formula : $w = 690 + 11{\cdot}5l = 690 + 11{\cdot}5 \times 68 = 1472$ pounds per ft., or 736 pounds on each main girder, for a single line of railway. The loads and stress will be expressed in units of 1000 pounds, and the moments in units of 1000 foot pounds.

The Live Load may be taken as Cooper's E_{50} locomotive, followed by a train of 5000 pounds per foot run. Moments and shears due to dead load : The reaction at the supports, or end shears, is : $0{\cdot}736 \times 34 = 25{\cdot}024$ (say 25) units. The shear at one-quarter points, or 17 ft. from the middle of supports, or the ends of the effective span, is 12·5 units. The shear at the centre is zero. The moment at the centre is—

$$\frac{wl^2}{8} = \frac{0{\cdot}736 \times \overline{68}^2}{8} = 429.$$

The moment at the quarter-points is—

$$25 \times 17 - \frac{0{\cdot}736}{2} \times \overline{17}^2 = 319.$$

Live Load.—See diagram, Fig. 383.

At one-quarter of the span the maximum moment occurs when the wheel 3 is placed at that point, and the length of the uniform load on the bridge is $13 + 51 - 53 = 11$ ft., and the total load on the bridge is $177{\cdot}5 + 11 \times 2{\cdot}5 = 205$. The criterion $\frac{l'}{l} = \frac{62{\cdot}5}{205}$ or $\frac{37{\cdot}5}{205}$ shows that the former is greater and the latter less than one-quarter, hence the wheel 3 is in the correct position.

The moment about the right support is—

$$\mathrm{W}y + \mathrm{W}x + \frac{wx^2}{2} = 4370 + 177{\cdot}5 \times 16 + 1{\cdot}25 \times \overline{11}^2 = 7361.$$

The moment at one-quarter points is—

$$\frac{7361}{4} - 287{\cdot}5 = 1552{\cdot}75 \text{ or } 1{,}552{,}750 \text{ foot pounds.}$$

At the centre the maximum moment occurs when wheel 5 is at this point. The uniform load $= 23 + 34 - 53 = 4$ ft. The total load on the bridge is—

$$177{\cdot}5 + 4 \times 2{\cdot}5 = 187{\cdot}5,$$

and $\frac{l'}{l} = \frac{112{\cdot}5}{187{\cdot}5}$ or $\frac{87{\cdot}5}{187{\cdot}5}$, the former being greater and the latter less than $\frac{1}{2}$. The moment about the right support is—

$$4370 + 177{\cdot}5 \times 9 + 1{\cdot}25 \times 4^2 = 5987{\cdot}5.$$

The moment at the centre is—

$$\frac{5987{\cdot}5}{2} - 1037{\cdot}5 = 1956{\cdot}25.$$

Impact Allowance for the Live Loads.—If we adopt Pritchard's formula, we have $\mathrm{I} = {\cdot}75l\left(\frac{l}{l + d}\right)$ where l, in the brackets, may be taken as the moment at the centre due to the live load, and d as the moment at the same point due to the dead load—

$$\mathrm{I} = {\cdot}75l \times \frac{1956}{2385} = 0{\cdot}615l, \text{ or } 61{\cdot}5 \text{ per cent.}$$

If we adopt the American Bridge Company's formula, $\mathrm{I} = \frac{300}{300 + \mathrm{L}}$, where L = the effective span, we have $\mathrm{I} = \frac{300}{368} = 82$ per cent.

We will adopt 80 per cent. increase in all the live load stresses; thus—

Total bending moment in the centre is—

$$\begin{aligned} d &= 429 \\ l &= 1956 \\ \mathrm{I} &= 1565 \\ \hline & 3950 \text{ foot pounds.} \end{aligned}$$

At one-quarter points the moment is—

$$\begin{aligned} d &= 319 \\ l &= 1553 \\ \mathrm{I} &= 1242 \\ \hline & 3114 \text{ foot pounds.} \end{aligned}$$

Shears due to Live Load.—The maximum shearing stress will occur when wheel 2 is placed at the centre of the left support, requiring 23 feet of the uniform load on the bridge. The moment of all the loads about the right support is—

$$3770 + 165 \times 28 + 1{\cdot}25 \times \overline{23}^2 = 9051{\cdot}5,$$

where 3770 = 4370 − 600 or the moment about wheel 9. The reaction at the left support is—

$$\mathrm{R} = \frac{9051{\cdot}5}{68} = 132{\cdot}8.$$

The shear at the quarter-points will be found by placing the wheel 2 at 17 ft. from the left end of the span, requiring 6 feet of the uniform load on the bridge, and producing a total load of—

$$177{\cdot}5 + 6 \times 2{\cdot}5 = 192{\cdot}5 \text{ units.}$$

The moment about the right support is—

$$4370 + 177{\cdot}5 \times 11 + 1{\cdot}25 \times 6^2 = 6367{\cdot}5.$$

The end reaction is therefore—

$$\frac{6367{\cdot}5}{68} = 93{\cdot}64 \text{ units.}$$

The shear at the quarter-points is therefore—

$$93{\cdot}64 - 12{\cdot}5 = 81{\cdot}14 \text{ units.}$$

The shear at the centre may be found in a similar manner to be 37·8 units.

The total shear at the ends is—

$$\begin{aligned} d &= 25{\cdot}0 \\ l &= 132{\cdot}8 \\ \mathrm{I} &= 106{\cdot}2 \\ \hline & 264{\cdot}0 \text{ units.} \end{aligned}$$

At the quarter-points—

$$\begin{aligned} d &= 12{\cdot}50 \\ l &= 81{\cdot}14 \\ \mathrm{I} &= 64{\cdot}77 \\ \hline & \quad 158{\cdot}41 \text{ units.} \end{aligned}$$

At the centre—

$$\begin{aligned} l &= 37{\cdot}8 \\ \mathrm{I} &= 30{\cdot}2 \\ \hline & \quad 68{\cdot}0 \text{ units.} \end{aligned}$$

The uniform load producing the same moment in the centre as the concentrated wheel loads and the train is—

$$\frac{wl^2}{8} = 1956 \quad \therefore\ w = 3{\cdot}382 \text{ units,}$$

or producing the same end shear as the wheel loads and train—

$$\frac{wl}{2} = 132{\cdot}5 \quad \therefore\ w = 3{\cdot}906 \text{ units.}$$

Flange Areas.—It is usual to design the tension flange, and to make the compression flange similar to it in every respect. The area of the flange in the centre is found by dividing the total moment by the depth of the girder, and by the working stress—

$$\frac{3950}{7 \times 17} = 33{\cdot}2 \text{ sq. in.}$$

This section must be made up of angles and flange plates, and it is sometimes the practice to consider that the bending moment is resisted partly by the web, allowing an eighth of its area as part of the flange area. If the web plate is $\frac{7}{16}$ in. thick, $\frac{1}{8}$ of its area is 4·6 sq. in. requiring 28·6 to be provided in plates and angles, and also necessitating special consideration in the design of the joints in the web to resist this bending moment in addition to the shearing stress. We will assume in this design that the whole of the bending is resisted by the flanges without the assistance of the web.

Flange Angles.—The angles uniting the web with the flange plates must be large enough to permit of two rows of rivets in the vertical legs, and two rows may frequently be necessary in the horizontal legs. The size of a rivet hole for a $\frac{7}{8}$ in. rivet is $\frac{15}{16}$ in., but in deducting the diameters of the holes to obtain the net sectional area, we will take 1 in. Assume that the angles are 6 in. × 6 in. × $\frac{5}{8}$ in. having the arrangement of rivet holes shown in Fig. 384. The net area of one angle after deducting two rivet holes is 5·86 sq. in., and for two angles 11·72 sq. in. The area required in the centre for the plates is—

$$33{\cdot}2 - 11{\cdot}72 = 21{\cdot}48 \text{ sq. in.}$$

The section may be made up as follows:—

1 plate 16 in. wide, $\frac{5}{8}$ in. thick =	8·75 sq. in.
2 plates 16 in. wide, $\frac{9}{16}$ in. thick =	15·75 ,,
	24·50 sq. in.

The arrangement of the plates is largely a matter of judgment, but the minimum thickness should not be less than ⅜ in., and the maximum thickness not greater than ¾ in. The thickest plate should be next to the angles, and should be carried through the entire length of the girders, but the remaining plates may be carried just far enough to resist the stresses developed at the sections in question. A parabola may be constructed with the central ordinate representing 33·2 sq. in. and the area of the plates provided also set off to the same scale as shown in Fig. 385. The plates are always extended about 18 in. beyond the points on each side found in the manner indicated. The lengths may also be calculated in the following manner:—

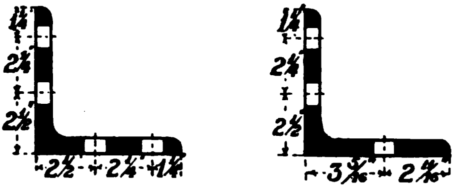


FIG. 384.

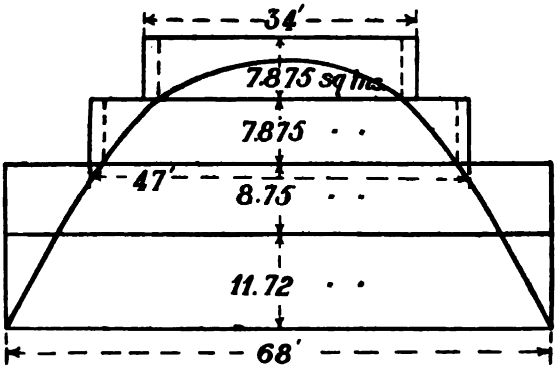


FIG. 385.

Let A = the total net flange area = 36·22 sq. in.
a = the net area of one plate.
l' = the length of the flange plate.
l = the effective span.

Then—

$$\frac{l'^2}{l^2} = \frac{a}{\mathrm{A}} \text{ and } l' = l\sqrt{\frac{a}{\mathrm{A}}}.$$

$$\text{For the top plate, } l' = 68\sqrt{\frac{7{\cdot}875}{36{\cdot}22}} = 31{\cdot}3 \text{ ft.}$$

$$\text{,, second ,, } l' = 68\sqrt{\frac{15{\cdot}75}{36{\cdot}22}} - 44{\cdot}9 \text{ ,,}$$

The actual lengths provided are 34 and 47 ft.

Web Thickness.—The end shear has been found to be 264 units, and allowing 10 units as the safe working stress in shear we find the area required is 26·4 sq. in. at the quarter-points, and in the centre the areas required are 15·8 and 6·8 sq. in. respectively. The web of a plate girder consists generally of one or more plates stiffened at intervals by means of angles or tees at a distance apart not exceeding 5 ft. and not exceeding the depth of the girder. Sometimes additional plates, or a thicker plate, is used at the ends for a short distance. In this case it would appear that a plate ⅜ in. thick would be capable of resisting the shearing stress, but in view of the tendency of the plate to buckle and other considerations, we will make it $\frac{7}{16}$ in. thick throughout the whole length.

Joints in Web Plate.—The number of joints or splices required in the web plate will depend upon the largest size of plate obtainable and the most convenient lengths suitable for handling in the shop, also as to whether the girder is to be sent to the site of the bridge in sections or in one piece. In America it is usual to complete the girders in the shop and send them by rail up to lengths of 80 and even 100 ft.; also much longer and larger plates are used than in Great Britain, so that the joints are reduced in number. In this case we will divide the length of 70 ft. into three portions—22 ft. 3 in., 25 ft. 5¾ in. and 22 ft. 3 in. The sum of these lengths is ¼ in. shorter than the total length of the girder, and the depth of the web plate is also made ¼ in. less than the depth measured from back to back of the main angles. There will be two joints in the web plate, and cover plates, or splice plates, will be arranged on each side of the plate. The number of rivets must be sufficient to transfer the stresses developed at the joint to the flanges, and it is usual to consider the rivets in the web covers between the flange angles only, and not those passing through the flange angles. The length of the web cover plates will be 72 in. The shearing stress at the joints, 21 ft. 3 in. from supports, is found thus: Place wheel 2 at the point over the web joint, then the length of uniform load on the bridge is 46·7 + 8 − 53 = 1·7′, and the total load is 177·5 + 1·7 × 2·5 = 181·75 units.

The moment about the right support is—

$$4370 + 177{\cdot}5 \times 6{\cdot}7 + 1{\cdot}25 \times \overline{1{\cdot}7}^2 = 5562{\cdot}86.$$

The reaction at left support $= \frac{5562{\cdot}86}{68} = 81{\cdot}8$. The shear $= 81{\cdot}8 - 12{\cdot}5 = 69{\cdot}3$ units. The total shear—

$$\begin{aligned} d &= 15{\cdot}6 \\ l &= 69{\cdot}3 \\ \mathrm{I} &= 55{\cdot}4 \\ \hline & 140{\cdot}3 \text{ units.} \end{aligned}$$

The number of rivets required is found by dividing the total shear by the value of a $\frac{7}{8}$ rivet. These rivets are in double shear, but the value of a rivet bearing on a $\frac{7}{16}$ plate is only 7656 pounds, so that this must govern the number required—

$$\frac{140{\cdot}3}{7{\cdot}656} = \text{over } 18.$$

The minimum pitch is therefore—

$$\frac{72}{18} = 4 \text{ in.}$$

Although the web is supposed to resist only the shearing stress, the strain near the flange angles, however, must be greater than near the neutral axis, and it is better practice to make the pitch of the rivets in the web plate joints smaller near the flange angles at the top and bottom of the joint cover plates in order to resist more effectively the bending action. It is also good practice to provide a larger number of rivets than would be necessary merely to resist the shearing stress. In the present case thirty-eight rivets are provided on each side of the joint, and the pitch of the first twelve rivets at the top and bottom is made 3 in.; the pitch for the rivets in the centre, 3 ft. $6\frac{1}{2}$ in., is made $4\frac{1}{4}$ in. The joint cover plates are 14 in. wide and $\frac{3}{8}$ in. thick, and the angle stiffeners, which will be considered specially, are crimped so as to fit over the flange angles.

The Rivets uniting the Web with the Flange Angles.—The main duty of the rivets uniting the web with the flange angles is to transfer the increment of flange stress between any two sections from the web to the flanges. The increment may be found by dividing the difference between the bending moments at the two sections by the depth of the girder, and this difference is obviously a maximum at the ends of the girder where one section is taken over the support. It follows, from the equality of the horizontal and vertical shearing stresses, that the total vertical shear at the ends of a girder must be transmitted to the main flange angles, through the horizontal rows of rivets, in a distance equal to the effective depth of the girder. The variation in flange stress for a horizontal distance equal to the pitch of the rivets must be transferred through the rivet uniting the vertical legs of the flange angles to the web. If S denote the total shear, d the effective depth of the girder, and r the value of a rivet in resisting shear or bearing pressure on the web plate, whichever is the least, then it can be easily shown that—

$$\frac{S}{d} = \frac{r}{p}, \text{ or } p = \frac{rd}{S},$$

where p denotes the pitch of the rivets. Since the bearing value of the rivet, 7656 pounds, is less than the resistance to double shear, it follows that this value will govern the number and pitch of the rivets. The pitches of the rivets at the ends, quarter-points and centre may be found as follows:—

At ends—

$$\frac{7{\cdot}656 \times 7 \times 12}{264} = 2{\cdot}43 \text{ in.}$$

At quarter-points—

$$\frac{7{\cdot}656 \times 7 \times 12}{158{\cdot}41} = 4 \text{ in.}$$

At the centre—

$$\frac{7{\cdot}656 \times 7 \times 12}{68} = 9{\cdot}4 \text{ in.}$$

The figures represent the maximum pitch allowable, provided that in no case the pitch exceeds 6 in. In a deck bridge the rivets in the vertical legs of the flange angles in the top flange sustain a direct stress, due to the concentrated wheel loads above them, in addition to the horizontal shearing stress. If R denotes the resultant stress, then—

$$R = \sqrt{\left(\frac{264 \times 2}{7}\right)^2 + 25^2} = 79 \text{ units.}$$

If we assume that the wheel loads are distributed over a length of at least 2 ft., the number of rivets in this length is—

$$\frac{79}{7{\cdot}65} = \text{over 10 rivets or } 2{\cdot}4 \text{ pitch.}$$

The reduction in the pitches for the quarter-points and the centre may be found in a similar manner. If the pitch at the ends is made about 2½ in. and increased to about 5 in. in the centre, the necessary rivet area will be provided. The actual arrangement of the rivets is shown on the drawing Plate I. The rivets through the horizontal flange plates may be arranged in two rows as shown on the drawing; their duty is to transfer the increment of flange stress as in the case of the rivets through the vertical legs of the angles. The rivets are in single shear, and the value of a rivet is 6000 pounds for a dead load stress, therefore the pitches of the rivets are—

At the ends—

$$\frac{6\cdot000 \times 7 \times 12 \times 2}{264} = 3\cdot8 \text{ in.}$$

At one-quarter points—

$$\frac{6\cdot000 \times 7 \times 12 \times 2}{158\cdot41} = \text{over } 6 \text{ in.}$$

At the centre—

$$\frac{6\cdot000 \times 7 \times 12 \times 2}{68} = \text{over } 14 \text{ in.}$$

The actual arrangement of the rivets is shown on the drawing Plate I.

The number of rivets at and near the ends of the two upper flange plates each 16 in. wide and $\frac{9}{16}$ in. thick must be sufficient to transfer to these plates the stress they are designed to carry. In this case the stress carried by one of these plates at the centre of its length in the net section is—

$$7\cdot875 \times 17 = 133\cdot8 \text{ units.}$$

At the ends of these plates the stress is less, and the number of rivets necessary to transfer the maximum stress of 133·8 units is—

$$\frac{133\cdot8}{6} = 22\cdot3, \text{ say } 24.$$

If the rivets are 3 in. pitch, and there are two rows, the length necessary to transfer the maximum stress is about 3 ft.

Web Stiffeners.—The web must be stiff enough to resist any tendency to buckle, and angle stiffeners are generally arranged at intervals of not more than 5 ft. on each side of the web plate designed to act as compression members resisting the whole of the vertical shear. It has already been shown in Chapter I. that the intensities of shearing stress are equal on two planes at right angles to each other, and that the intensity of shearing stress in a rectangular beam is three halves the mean at the neutral axis, where it is a maximum, diminishing to zero at the top and bottom of the beam. The lines of maximum tensile and compressive stress intersect the natural axis at an angle of 45 degrees. We may imagine a network of these lines to be distant 1 in. apart, to fix the idea, dividing the web plate into a series of elementary columns equal in length to a line inclined at 45 degrees between the flange angles, if there are no stiffeners, or between the vertical stiffeners, if stiffeners are used. The compressive stress tends to produce buckling, and the tensile stress to resist it more or less, so that we may consider the total vertical shear distributed uniformly over the depth of the girder and loading each elementary column. If $\frac{l}{r}$ denotes the ratio of the length to the least radius of gyration, we may apply any of the column formulæ to determine the stress producing buckling, but from the nature of the case and the impossibility of expressing the actual conditions in a simple manner, a straight line formula is preferable to the more exact formulæ discussed in Chapter XII.

In Cooper's specifications: "The webs are required to be stiffened at intervals, not exceeding the depth of the girder, or a maximum of 5 ft. wherever the shearing stress per sq. in. exceeds the stress allowed by the following formula—

$$\text{Allowed stress} = 10{,}000 - 75H.$$

Where H = the ratio of the depth of the web to its thickness." Another rule is to introduce stiffeners in the manner described whenever the web thickness is less than sixty times the unsupported length between the flange angles.

In the present case the actual shearing stress is—

$$\frac{264}{36{\cdot}75} = 7{\cdot}2 \text{ units.}$$

The web is quite unable to resist the tendency to buckle without the aid of stiffeners. The stiffeners must be designed to carry the maximum vertical shear, and it would be advantageous to arrange them at 45°, if it were not for the practical inconvenience of such an arrangement. Cooper's formula for stiffeners is—

$$\text{Allowed stress} = 10{,}000 - 45\frac{l}{r}.$$

Where l = length, and r = the least radius of gyration. The dead load column formula is given by Cooper as—

$$\text{Allowed stress} = 20{,}000 - 90\frac{l}{r}.$$

The area of the stiffener is found by dividing the vertical shear by the allowed stress.

The area required in the end stiffeners may be provided by means of four angles each 5 in. × 3½ in. × ⅜ in., together with two fillers 10 in. × ⅝ in. each, giving a total area of 25·84 sq. in. The actual stress is—

$$\frac{264}{25{\cdot}84} = 10{\cdot}2 \text{ units.}$$

The allowed stress is about 16 units (see Fig. 386 and drawing Plate I.). Hence this arrangement makes a very efficient stiffener capable of resisting the hammering effect at the ends of the girder due to trains at high

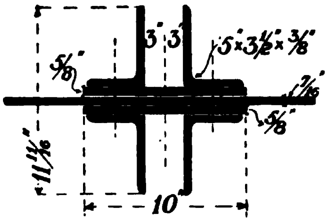


FIG. 386.

speed. The rivets required will be governed by the bearing value, 7·656 units, so that—

$$\frac{264}{7·656} = \text{over } 34.$$

The arrangement provides three rows of nineteen each, or fifty-seven rivets, the middle row passing through the fillers and web plate only. The spacing of the rivets is closer at the top and bottom of the stiffeners near the flange angles as in the web joint plates. The arrangement of the intermediate stiffeners is largely a matter of judgment. They are generally spaced about 5 ft. centre to centre in the middle portion of the girder, and 3 to 4 ft. centres near the ends. They may be constructed with two angles, 5 in. × 3½ in. × ⅜ in., and one filler, 3½ in. wide by ⅜ in. thick, providing an area of about 11 sq. in. For the arrangement of the stiffeners in the design, see drawing, Plate I.

Joints in Angles and Flange Plates.—In this design the main angles and the flange plates are shown in one length, so that there are no joints or splices to be considered. It is, however, frequently necessary to divide the girder into two or more sections for convenience of shipment, even where there are facilities for handling the full lengths of the plates and angles in the shop, necessitating joints both in the plates and angles. The method of designing such joints has been fully considered in Chapter XIII., and need not be further dealt with here.

Web Joint Plates or Splices Resisting Bending.—In this design it has been assumed that the bending moment is resisted entirely by the flanges and the shearing stresses by the web. As the web must necessarily assist the flanges in resisting bending, it is sometimes the practice to assume that one-eighth of the gross area of the web plate is concentrated in the flanges. In this case the web joint plates must be designed to resist both bending and shear, and horizontal plates are arranged at the top and bottom of the web (as shown in Fig. 387) of sufficient size to provide the necessary rivet area to resist the bending moment, leaving the vertical joint plates to resist the shearing stress only. In this case the area of the web plate is 84 in. × $\frac{7}{16}$ in. or 36·75 sq. in., and one-eighth of this is 4·6 sq. in. The stress transferred to the flanges, allowing 17 units per sq. in., is—

$$17 \times 4·6 = 78·2 \text{ units.}$$

The number of rivets required in the horizontal plates at the top and the bottom, allowing the bearing value on the $\frac{7}{16}$ in. plate, 7·656 units, is—

$$\frac{78·2}{7·656} = \text{over 10, say 12 rivets.}$$

The cover plates in Fig. 387, providing twelve rivets on each side, 3 in. pitch, are 24 in. long and 9 in. deep, leaving 54 in. for the vertical plates, which must contain the full number of rivets to resist the maximum shear. The arrangement of the web joint designed to resist bending, as well as shear, is shown in Fig. 387.

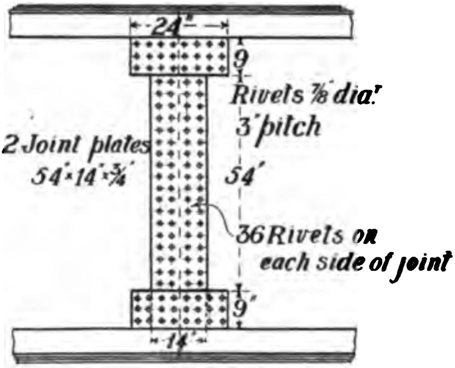


FIG. 387.

Lateral Bracing.—The subject of lateral bracing has been treated in Chapter XIV. In this case the bracing consists of an upper lateral system of L-bars, each 6 in. × 4 in. × ½, arranged as shown on Plate I. There is no bottom lateral system provided in this design, so that the top system must resist the whole of the wind pressure acting on the exposed area of the girder up to the level of the rails, together with that on a moving train. In girders of 80 ft. span it would be preferable to arrange a top and bottom lateral system and to divide the wind pressure on the girder and deck between them equally, leaving the top lateral system to resist the effect of the wind load on the train in addition to half that on the girder and deck. It is also necessary to provide frames in a vertical plane at the ends and at four intermediate points to act as sway bracing resisting the effect of the wind pressure and vibrations due to the live load. The use of angles in the tension, as well as in the compression members, is good practice, as they resist vibrations much better than flat bars; but in this case all the bars may be in compression. For the sake of simplicity in the design, and for practical considerations, it is better to make the bars in the top lateral system of the same section throughout, so that in this case only the end bar need be considered as resisting both tensile and compressive stresses. It will be sufficient to provide for wind pressure and vibrations if we allow 300 pounds per lineal ft., treated as a dead load, and 450 pounds per lineal ft., treated as a live load, producing loads at the panel points of 3·075 and 4·612 units respectively at five of the panel points, also 2·163 and 3·224 units respectively at the panel point nearest the end on the left. It can be easily shown that the maximum reaction on the left is 29·8 units and the stress in the end lateral bars 36·17 units. The area required in tension, allowing Cooper's stress of eighteen units, is about 2 sq. in., and in compression, using Cooper's formula for the allowed stress, $\sigma = 13000 - 60\frac{l}{r}$, where $\frac{l}{r} = 80$ about, and $\sigma = 8200$ pounds, or 8·2 units: hence the area in compression $= \frac{36 \cdot 17}{8 \cdot 2} = 4 \cdot 5$ sq. in.

The 6 in. × 4 in. × ½ in. angles provide a gross area of 4·75 sq. in. and a net area, neglecting the shorter leg of the angle, of 2·5 sq. in., so that they are sufficient to resist the stresses at the ends and also throughout the girder. The arrangement of the frames in a vertical plane is shown in Plate I., and it can be shown that the released load on the wind side, and the increased load on the lee side, is about 30 units at the end frames, requiring about 2 sq. in. for tensile stress, or 3·7 sq. in. for compressive stress; but since the diagonal bars are crossed, and consist of angles 6 in. × 4 in. × ⅜ in., the area provided is ample. It can also be shown that the angles in the intermediate frames, 3½ in. × 3½ in. × ⅜ in., are in like manner sufficient to resist the stresses developed in them. The horizontal members of the end and intermediate frames may be made of the same sections of angles used for the diagonals although the stresses in them are a little less.

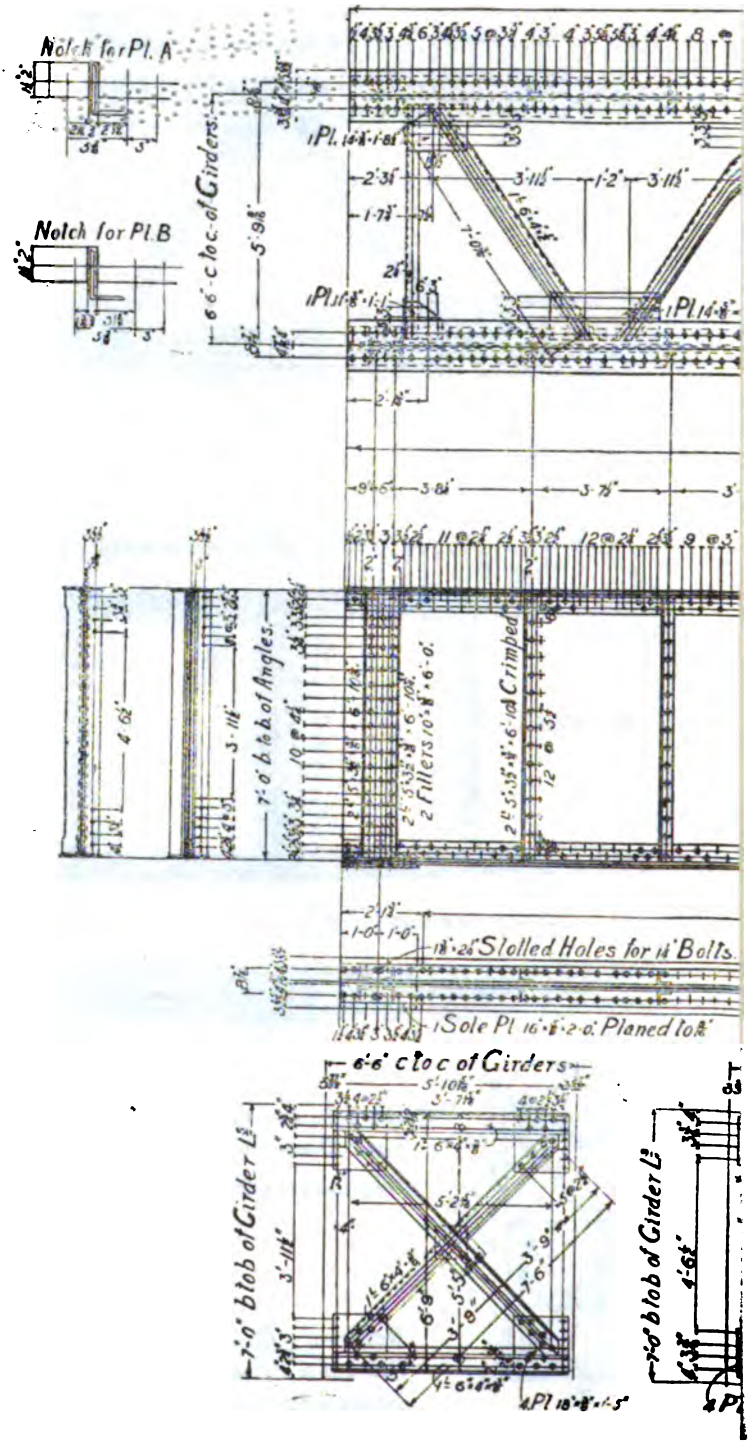
The weight of the bridge may now be determined and the bill of quantities arranged in the following manner :—

BILL OF QUANTITIES.

		Weight in pounds.
Two flange plates each 34′ 1″ × 16″ × $\frac{9}{16}$″ = 34′ 1″ × 30·6 × 2	=	2085·9
,, ,, 47′ 10″ × 16″ × $\frac{9}{16}$″ = 47′ 10″ × 30·6 × 2	=	2927·4
,, ,, 70′ 0″ × 16″ × ⅝″ = 70′ × 34 × 2	=	4760·0
Four angles each 70′ 0″ × 6″ × 6″ × ⅝″ = 70′ × 24·17 × 4	=	6767·6
Web plate 70′ 0″ × 7′ 0″ × $\frac{7}{16}$″ = 70′ × 17·85 × 7	=	8746·5
End stiffeners each 4 × 6′ 10¾″ × 5″ × 3½″ × ⅜″ = 55·2′ × 10·36	=	571·9
,, fillers each 2 × 6′ 0″ × 10″ × ⅝″ = 24′ × 21·25	=	510·0
Intermediate stiffness 26 × 6′ 10⅜″ × 5″ × 3½″ × ⅜″ = 179·3′ × 10·36	=	1857·5
,, fillers 22 × 6′ 0″ × 3½″ × ⅝″ = 132′ × 7·44	=	982·0
Web joint covers 4 × 6′ 0″ × 14″ × ⅜″ = 24′ × 17·85	=	428·4
		29,637·2
Two girders each 29,637·2 pounds	=	59,274·4

Lateral Bracing.

13 angles each 7′ 8″ × 6″ × 4″ × ½″ = 99·7′ × 16·16	=	1611·2
12 joint plates 3′ 0″ × 14″ × ⅜″ = 36 × 17·86	=	643·0
2 ,, 1′ 8½″ × 14″ × ⅜″ = 3·4 × 17·86	=	61·0
4 ,, 11½″ × 11″ × ⅜″ = 3·9 × 14·03	=	55·0
2 ,, 1′ 1″ × 11″ × ⅜″ = 2·2 × 14·03	=	31·0
13 washers 3″ × 4½″ × ¼″ = 13 × 0·93	=	12·0
		2413·2



Notch for Pl. A
Notch for Pl. B
6'-6" c to c of Girders
5'-9⅝"
1 Pl. 14·⅜·1-8
1 Pl. 11·⅜·1-1
1 Pl. 14·⅜
3'-11½"
1'-2"
3'-11½"
2'-3⅛"
1'-7⅞"
2'-1⅛"
3'-8⅛"
3'-7½"
11 @ 2⅛"
12 @ 2⅛"
9 @ 3"
7'-0" b to b of Angles.
10 @ 4⅛"
4'-6⅞"
3'-11⅞"
2 Fillers 10"·⅜"·6'-0"
2 L's 5·3½·⅜·6'-10½" Crimped
12 @ 3½"
2'-1⅛"
1-0
1-0
1⅛·2⅛" Slotted Holes for ⅞" Bolts.
1 Sole Pl. 16"·⅜·2-0. Planed to ⅞"
6'-6" c to c of Girders
5'-10⅞"
5'-7⅞"
5'-2⅞"
7'-0" b to b of Girder L's
3'-11⅞"
1 L 6"·4·⅜
3'-9"
7'-6"
4 Pl. 18"·⅜"·1-5"
7'-0" b to b of Girder L's
4'-6⅞"

Cross Frames.

	Weight in pounds.
Ends—	
4 angles 6′ 1$\frac{5}{16}$″ × 6″ × 4″ × $\frac{3}{8}$″ = 24·45 × 12·28	300
4 „ 7′ 11$\frac{1}{4}$″ × 6″ × 4″ × $\frac{3}{8}$″ = 31·8 × 12·28	390
8 plates 1′ 5″ × 18″ × $\frac{3}{8}$″ = 2·1 × 15 × 8	252
2 „ 10″ × 10″ × $\frac{3}{8}$″ = 1·7 × 12·75	22
	973
Intermediate—	
8 angles 6′ 1$\frac{5}{16}$″ × 3$\frac{1}{2}$″ × 3$\frac{1}{2}$″ × $\frac{3}{8}$″ = 48·9 × 8·45	413
8 „ 7′ 10$\frac{9}{16}$″ × 3$\frac{1}{2}$″ × 3$\frac{1}{2}$″ × $\frac{3}{8}$″ = 63 × 8·45	532
16 plates 1′ 2$\frac{1}{2}$″ × 15″ × $\frac{3}{8}$″ = 19 × 19·13	370
4 „ 8″ × 8″ × $\frac{3}{8}$″ = 2·8 × 10·2	29
	1344

The weights of the various portions of the bridge are as follows :—

2 main girders = 59,274 pounds.
Lateral bracing = 2,413 „
Sway bracing = 2,317 „

64,004 „
Add 3½ per cent. for rivet heads = 2,240 „

Total weight of steel in bridge = 66,244 „
Weight per lineal ft.—

$$\frac{66244}{70} = 946{\cdot}3 \text{ pounds.}$$

Add for the deck rails, fastenings, etc. = 500 „

Total weight of bridge = 1446 „ per foot run.

The assumed weight obtained from the formula $w = 690 + 11{\cdot}5l = 1472$ pounds, which is as nearly correct as could be expected.

The dead load in this bridge is small in consequence of the use of the open timber deck, but for any other type of deck it will generally be much greater. In this bridge the main girders are fixed at one end, and are free to expand at the other end by sliding on planed surfaces, one of which should be of gun metal. The sliding plates are provided with elongated holes, and the bolts with ferrules, so that the nuts do not press on the plates and impede their movement.

For girders over 80 ft. in span, the ends should be provided with saddles, prepared to rest on a turned pin, in order that the distribution of pressure over the end bearings may be uniform. The expansion end should be provided with rollers instead of sliding plates. The expansion bearings shown on Plate II.*b* are suitable for a long span.

CHAPTER XVIII.

DESIGN OF A PIN-CONNECTED PRATT TRUSS.

In this example it is proposed to illustrate the design of a pin-connected truss bridge as used in America (Plates II. and II.*a*). It is usual to design all such bridges in accordance with a specification written by the engineer, or to adopt the general specifications of Mr. Theodore Cooper, or those of the bridge company in question. The actual design as illustrated in Plate II.*a* represents the American Bridge Company's standard 150 ft. span for a single line of way, but all the calculations have been made by the writer independently. The figures written on the outline drawing (Plate II.) correspond so closely with those obtained by the bridge company that no modifications have been made in the sectional area and design of the various members of the bridge, which may be taken as a good example of the best American practice for this type.

The effective depth of the trusses measured from centre to centre of the pins in the top and bottom chords is 28 ft., the panel length is 25 ft., and the distance between the centres of the two trusses, 16 ft. 2 in. The weight of the rails, fastenings and the open timber deck has been taken at 400 pounds per lineal ft., and the live load is taken as represented by two of Cooper's locomotives E_{50} followed by a train equivalent to a uniform load of 5000 pounds per ft. run, or a special load equally distributed on two pairs of driving wheels spaced 6 ft. centre to centre of 120,000 pounds. The allowance for wind and vibrations is taken in accordance with Cooper's specifications. The material used is medium steel excepting in the rivets.

Stringers and Cross-girders.

The deck will consist of cross-sleepers of white oak, yellow pine, Oregon or other approved timber 8 × 8 in. cross-section with every fourth sleeper bolted to the top flange of the stringer, also there will be two timber guard rails bolted to every fourth sleeper each 5 × 8 in. cross-section. The stringers will consist of plate web girders, 25 ft. span, the cross-girders or floor beams will also consist of plate web girders of 16 ft. span. The economic depth of the stringer is about one-eighth of the span, and will be taken as 3·15 ft. or 37·8 in. In order that the stringers may be riveted to the web of the cross-girder, between the main flange angles, the total depth of the cross-girder is made 4 ft. 4¼ in., and of the stringer 3 ft. 5½ in.

Dead Load on Stringer.—The dead load consists of the rails, fastenings, sleepers, and guard rails, also of the weight of the stringer and wind-bracing—

$$\begin{aligned} &\text{deck, } 25 \times 200 &&= 5000 \text{ pounds} \\ &\text{girder and half-bracing} &&= 4500 \quad ,, \end{aligned}$$

since $wl = al^2 = 7\cdot1 \times \overline{25}^2 = 4437\cdot5$.

The total dead load is 9500 pounds, or 9·5 units.

The maximum end shear = 4·75 units, and the maximum moment $= \dfrac{9\cdot5 \times 25}{8} = 29\cdot7$ units.

Live Load on Stringer.—The maximum end shear occurs when the second wheel of the locomotive is placed at the left end, and the sixth wheel one foot from the right end—

$$\mathrm{R} = \frac{16\cdot25}{25} + \frac{25}{25}(10 + 15 + 20 + 25) = 70\cdot65 \text{ units.}$$

The maximum moment occurs when the four driving wheels have their centre of gravity in the centre of the span—

$$\mathrm{M} = 50 \times 12\cdot5 - 25(7\cdot5 + 2\cdot5) = 375 \text{ units.}$$

The allowance for impact may be found from the formula—

$$\mathrm{I} = l\left(\frac{l}{l + d}\right) = l\left(\frac{375}{404\cdot7}\right) = \cdot91l \text{ for moments,}$$

$$\text{and } \mathrm{I} = l\left(\frac{70\cdot65}{75\cdot4}\right) = \cdot937l \text{ for end shears.}$$

These are nearly the same as the values found by Mr. Schneider's formula, $\dfrac{l}{l + 300}$.

Adopting 94 per cent. increment for live load stresses, we have the equivalent dead load shear and moment—

$$\text{Maximum shear } = \begin{cases} 4\cdot75d \\ 70\cdot65l = 141\cdot81 \text{ units.} \\ 66\cdot41\mathrm{I} \end{cases}$$

$$\text{Maximum moment } = \begin{cases} 29\cdot7d \\ 375\cdot0l = 757\cdot2 \quad ,, \\ 352\cdot5\mathrm{I} \end{cases}$$

The area of flanges required in tension is—

$$\frac{757\cdot2}{3\cdot15 \times 17} = 14\cdot1 \text{ sq. in.}$$

Two angles 6 in. × 6 in. × $\frac{9}{16}$ in. provide a net area of 11·8 sq. in. and a gross area of 12·8 sq. in. Allowing 10 units stress in the web we require 14·13 sq. in., which is provided for by making the web ½ in.

thick, giving also an extra 4·8 sq. in. to assist in resisting the tendency to buckle. One-eighth of the gross area of the web, 2·6 sq. in., added to the net area of the two angles gives an effective flange area of 11·8 + 2·6 = 14·4 sq. in. It can easily be shown by Cooper's formula, or otherwise, that there will be no need of stiffeners.

The pitch of the rivets uniting the flange angles with the web is found thus—

$$p = \frac{rd}{S}, \text{ where } r = 10 \text{ units, } d = 37·8$$

$$p = \frac{10 \times 37·8}{142} = 2·6 \text{ in.}$$

The rivets provided are in two rows—2½ in. pitch for the 5 ft. nearest the cross-girders, 3 in. pitch for the next 3 ft. 9 in., and 4 in. pitch in the middle portion of the stringer. The end connection to the cross-girder is made with angles, 5 ft. × 3½ × $\frac{7}{16}$, and the number of field rivets required is—

$$\frac{142}{6} = 23·6.$$

In the design, Plate II.*a*, twenty-eight rivets are provided, also two fillers, each 7 in. × $\frac{9}{16}$ in., riveted to the web plate of the stringer with five rivets independently; also there are eleven rivets passing through the end angles, fillers and web of the stringer. These sixteen rivets are in double shear.

Wind Bracing.—The wind bracing between the two stringers will be arranged with angles capable of resisting both tension and compression, riveted through one leg to the upper flange angles. The cross-sleepers spaced 12 in. centres will offer considerable lateral resistance to the wind pressure and vibrations from train loads. The usual allowance for wind pressure is 600 pounds per lineal ft., 450 pounds being taken as a moving load and 150 as a fixed load; but since the angles are usually arranged as shown in Plate II.*a*, and are made of the same section throughout, we may consider each of the four apices loaded with three units, giving the end reaction six units and stress in diagonal bar about eight units. The allowed stress in compression, $13000 - 60\frac{l}{r}$, is about 4600 pounds per sq. in., or 4·6 units. The areas required are—

$$\frac{8}{4·6} = 1·74 \text{ sq. in. compression.}$$

$$\frac{8}{17} = 0·47 \quad ,, \quad \text{tension.}$$

The angles adopted are 3½ in. × 3½ in. × $\frac{5}{16}$ in., which provide a gross area of 2 sq. in. and a net area of 0·5 sq. in. in one leg of the angle.

The released load on the stringer at the windward side and the increased load on the lee side need not be considered in this case.

The weight of the stringer may be calculated in the following manner :—

	Pounds.
Four main angles, 24′ 11½″, at 21·88 pounds per ft. . . . =	2184
Web plate, 25′ × 41″ × ½″, at 42·5 pounds per ft. . . . =	1742
End angles, 13′ 4″ × 5″ × 3½″ × 7/16″, at 12·08 pounds per ft. . =	162
Four fillers, 2′ 5¼″ × 7″ × 9/16″, at 13·39 pounds per ft. . . =	130
Wind bracing, 20′ × 3½″ × 3½″ × 5/16″, at 7·1 pounds per ft. . =	142
	4360
Add 3 per cent. for rivet heads . . . =	131
Total weight of stringer . . . =	4491

The weight assumed was 4500 pounds.

Cross-girders.—The dead load upon a cross-girder is found from the dead load reactions of the stringers applied at points 3 ft. 3 in. from the centre, added to the weight of the cross-girder. Assume that the cross-girder weighs 250 pounds per lineal ft., or a total weight of 4000 pounds, equal to four units. The end shear is—

$$9·5 + 2 = 11·5 \text{ units.}$$

The moment in the centre is equal to forty-nine units.

Live Load.—The position of the wheel loads producing the maximum reaction of the stringers on the cross-girder can easily be shown to be as illustrated in Fig. 116. Wheel 4 is over the cross-girder under consideration. The reaction for the loads to the left is—

$$\frac{12·5}{25} \times 7 + \frac{25}{25}(15 + 20) = 38·5 \text{ units.}$$

The reactions for the loads to the right is—

$$\frac{16·25}{25}(6 + 11) + \frac{25}{25} \times 20 = 31·05 \text{ units.}$$

The total reaction is—

$$38·5 + 31·05 + 25 = 94·56 \text{ (say 95) units.}$$

The maximum shear is equal to 95 units. The impact effect is about 90 per cent. of this, or 85·5 units. The maximum moment for the live load is—

$$180·5 \times 4·75 = 857·375 \text{ units.}$$

The total maximum shear is—

$$11·5 + 180·5 = 192 \text{ units.}$$

The total maximum moment is—

$$857·375 + 49 = 906·375 \text{ units.}$$

The flange area required is—

$$\frac{906{\cdot}4}{17 \times 4{\cdot}18} = 12{\cdot}8 \text{ sq. in. (about).}$$

Two angles, 6 in. × 4 in. × $\frac{9}{16}$ in., provide a gross area of 10·6 sq. in. and a net area of 9·5 sq. in. The effective net flange area is found by adding one-eighth of the gross area of the web, or $\frac{52 \times \frac{1}{2}}{8} = 3{\cdot}3$ sq. in.—

$$9{\cdot}5 + 3{\cdot}3 = 12{\cdot}8 \text{ sq. in.}$$

The cross-girders must be securely attached to the main trusses, and the detail drawing of this connection should be carefully studied (Plate II.*a*). The number of rivets provided in the end connections is 36, and the minimum number required is—

$$\frac{192}{6} = 32.$$

The pitch of the rivets uniting the web with the flange angles is found in the following manner :—

$$p = \frac{rd}{S} = \frac{10 \times 50{\cdot}2}{192} = 2{\cdot}6.$$

The number provided between the compression members of the main trusses and the attachment of the stringers is seventeen.

It will be observed that the web is reinforced at each end, and the thickness of the web plate between the reinforced ends is ½ in., so that there will be no tendency to buckle. The actual weight of the cross-girder as designed in Plate II.*a* can easily be shown to be about 4300 pounds, but since a considerable portion of this weight is concentrated at the ends the assumed weight of 4000 pounds uniformly distributed is on the safe side.

Main Trusses.—The weight of the main trusses may be found from the formula—

$$w = al = 7{\cdot}2 \times 150 = 1080 \text{ pounds per ft.,}$$

or by the formula proposed by Messrs. Waddell and Hendrick, the dead load per lineal ft. is—

$$\begin{aligned} w = 8{\cdot}63(l + 1{\cdot}3\text{W} - 140) &= 8{\cdot}63(150 + 1{\cdot}3 \times 177{\cdot}5 - 140) \\ &= 2078 \text{ pounds.} \end{aligned}$$

The panel point load is therefore—

$$\frac{2078 \times 25}{2} = 25975 \text{ pounds for each truss.}$$

The actual weight of the trusses was known to be 1060 pounds per lineal ft., so that the panel point dead load may be estimated as follows :—

Deck = 200 × 25 = 5000 pounds
Floor = 2150 + 4500 = 6650 ,,
Truss = 530 × 25 = 13250 ,,

Total dead load at panel = 24900 ,,

Assume 25 units as the total dead load at the panel points, and take 7 units as acting at the top apices and 18 at the bottom: the value of $\sec \theta$ is 1·341, and of $\tan \theta$, 0·893. The dead load stresses are written on the outline of the truss shown in Fig. 221 and in Plate II.

Live Load Stresses.—The method of calculating the stresses due to the live load on the truss in order to produce the maximum shear and the maximum moment has already been considered in connection with the diagram for the concentrated load system shown in Fig. 283 and described in Chapter XVI. The same diagram and method will be used in order to complete the determination of the live load stresses in the various members of the truss.

Stresses in the Inclined and Vertical Web Members.—The work will be greatly facilitated by means of a tracing of the truss, which can be moved in such a manner as to bring any wheel load over a particular apex of the truss.

Member *oc*. Arrange the diagram and the tracing of the truss so that the wheel number 4 is immediately over the point *c*. Then since the criterion for maximum shear in the panel *bc* is satisfied for this position, the total load on the bridge is 377·5 and the length of the uniform load is 9 ft.

The moment about $g = 20{,}455 + 355 \times 9 + 1{\cdot}25 \times 81$
$= 23{,}761$

The shear in the panel *bc* is—

$$\frac{23761 - 6 \times 600}{150} = 134{\cdot}4.$$

The stress in *oc* = 134·4 × 1·341 = 180·23 units.

Member *nd*. The criterion is satisfied when the wheel 3 is at *d*, and the shear in the panel has already been found to be 78·6 units. The stress in *nd* is—

$$78{\cdot}6 \times 1{\cdot}341 = 105{\cdot}4 \text{ units.}$$

Counter braces *me* and *mc*. The criterion is satisfied when the wheel 3 is at *e*; the total load on the bridge is 190, and wheel 10 is 7 ft. from *g*.

The moment about $g = 5790 + 190 \times 7 = 7120.$

The shear in the panel *de* is—

$$\frac{7120 - 6 \times 287{\cdot}5}{150} = 35{\cdot}97 \text{ units.}$$

The stress in the counter braces is—

$$35{\cdot}97 \times 1{\cdot}341 = 48{\cdot}23 \text{ units.}$$

The stress in the member *dm* = 35·97 „

Member *eh*. The criterion is satisfied when wheel 2 is placed at *f*.

The moment at $g = 2050 + 128{\cdot}75 = 2178{\cdot}75$.

The shear in the panel *ef* is—

$$\frac{2178{\cdot}75 - 6 \times 100}{150} = 10{\cdot}5 \text{ units.}$$

The stress in *eh* is—

$$10{\cdot}5 \times 1{\cdot}341 = 14{\cdot}08 \text{ units.}$$

The stress found for *eh* shows that the bar *oc* must be designed for a tensile stress of 180·23 units, and a compressive stress of 14·08 units.

The stress in the bar *ob* will occur from the reaction of the cross-girders attached to this bar. The maximum reaction of the cross-girder has been found to be 95 units.

The stress in the bar *ob* cannot be greater than 95 units.

Member *ao*. The maximum stress in *ao* occurs for the maximum reaction at *a* resulting from the maximum shear in the panel *ab*. If the wheel 4 is placed at *b*, the total load on the truss is 440 units, and there is 34 ft. of uniform load. The moment about *g* is—

$$20{,}455 + 355 \times 34 + 1{\cdot}25 \times \overline{34}^2 = 33{,}970 \text{ units.}$$

The shear in the panel *ab* is—

$$\frac{33970 - 6 \times 600}{150} = 202{\cdot}47 \text{ units.}$$

The stress in the bar *ao* = 202·47 × 1·341.
= 271·5 units.

Stresses in the Horizontal Members.

Member *ab* and *bc*. The maximum stress in the members *ab* and *bc* will occur from the maximum shear in the panel *ab* multiplied by tan θ.

The maximum shear has been found to be 202·47 units, therefore the stress in each of the bars *ab* and *bc* is—

$$202{\cdot}47 \times 0{\cdot}893 = 180{\cdot}8 \text{ units.}$$

Member *on*. The centre of moments for the stress in *on* is the point *c*. $l' = 50$ and $\frac{l'}{l} = \frac{1}{3}$. The criterion is satisfied when wheel 8 is placed at the point *c*, and the total load on the truss is 440 units extending over 143 ft. including 34 ft. of uniform load—

$$\frac{l'}{l} = \frac{145}{440} \text{ or } \frac{161{\cdot}25}{440}.$$

The former is greater and the latter less than $\frac{1}{3}$, hence by moving the loads to the right or the left of *c*, the value of $\frac{l'}{l}$ becomes less or greater than $\frac{1}{3}$, so that the position assumed is correct.

$$\frac{M_w l'}{l} - M_p = \frac{33970 \cdot 0}{3} - 3563 \cdot 75$$
$$= 7759 \cdot 6 \text{ units.}$$

The stress in the member *on* is—

$$\frac{7759 \cdot 6}{28} = 277 \cdot 13 \text{ units (compression).}$$

The same result would obviously be obtained for the member *cd*, but the stress is tension.

Member *nm*. The centre of moments is the point *d*, and the maximum moment has been found (Chapter XVI.) to be 8805 units and the stress 314·464 units.

Final Stresses.—In order to determine the final stresses it is necessary in the first place to determine the percentage of the live load stress which must be added to the calculated live load stress to represent the impact effect. The impact stress will vary according to the method adopted, and the following table has been prepared in order to show the difference in the methods proposed by Pritchard and Schneider. In the former the coefficient *c* has been taken at 75 and at 90.

TABLE LXXI.

Member.	Dead load stress.	Live load stress.	Ratio $\frac{l}{l+d}$	$I = \cdot 75 \frac{l}{l+d}$	$I = \cdot 90 \frac{l}{l+d}$	Loaded length in ft. L.	$I = \frac{300}{L+300}$	Impact stress adopted $I = \cdot 90 l \left(\frac{l}{l+d}\right)$
ao or *gh*	− 84·0	− 271·5	·760	·57	·68	150	·67	− 184·28
ob ,, *hf*	18·0	95·0	·840	·63	·76	50	·86	72·28
ne ,, *ke*	− 19·5	− 78·6	·800	·60	·72	75	·80	− 56·59
md	− 7·0	− 36·0	·837	·63	·75	50	·86	− 27·00
oc or *he*	50·3	180·2 − 14·08	·734	·55	·66	100	·75	118·05
nd ,, *kl*	16·8	105·4	·862	·65	·78	75	·80	82·21
on ,, *kh*	− 89·3	− 278·0	·756	·57	·68	150	·67	− 189·00
nm ,, *mk*	− 100·4	− 314·5	·750	·56	·68	150	·67	− 213·86
ab ,, *bc* *ef* ,, *fg*	55·0	271·5	·830	·62	·75	150	·67	203·63
cd ,, *de*	89·3	278·0	·760	·58	·68	150	·67	189·00
me ,, *mc*	0	48·3	1·000	·75	·90	50	·86	43·47

It will be seen that Pritchard's formula with a coefficient of 90 gives results which do not differ very much from those obtained by Schneider's formula. The former will be adopted. The final stresses are written on the diagram Plate II. The design of the various details is shown in Plate II.*a*.

Wind Bracing.—The top and bottom lateral systems are illustrated in Plate II. The top system consists of two diagonals for each panel, but only one system can be in operation at the same time for the wind acting as shown, the other system is in operation when the wind acts in the contrary direction. The maximum stresses in the top system will occur with a wind force of 50 pounds per sq. ft., and the moving train load could not be on the bridge. The stresses in both the diagonals and horizontal compression members are written on the outline drawing Plate II., but the stress on the compression members will be greater when considered as part of the sway bracing than as part of the top lateral system.

The exposed area of the bridge is taken at 320 sq. ft. for each panel divided between the top and bottom panel points in the proportion of 120 and 200 respectively. The panel point pressures will be taken as equally divided between the joints on the wind and lee sides, thus—

$$\text{Top panel points} = \frac{50 \times 120}{2} = 3000 \text{ pounds} = 3 \quad \text{units.}$$

$$\text{Bottom panel points} = \frac{30 \times 200}{2} = 3000 \quad ,, \quad = 3 \quad ,,$$

$$\left.\begin{array}{r}\text{Moving load on bottom}\\ \text{panel points}\end{array}\right\} = \frac{500 \times 25}{2} = 6250 \quad ,, \quad = 6{\cdot}25 \quad ,,$$

The stresses in the two diagonals in the first and second panels counting from the centre are $3 \times \sec\theta$ and $9 \times \sec\theta$; since $\sec\theta = 1{\cdot}8$, these stresses are 5·4 and 16·2 respectively. Two angles each 3½ in. × 3½ in. × ⅜ in. are used, giving a net area in the leg riveted to the connecting plates of 2·06 sq. in., which is ample. The diagonals are riveted together where they intersect in the centre of the panel in the manner shown on Plate II.*a* in order more effectively to resist vibrations.

In the bottom lateral system each diagonal member is taken as resisting both tension and compression; the floor beams, however, are not considered as part of the lateral system. The diagonal members are riveted together where they intersect in the centre of each panel, and also at their intersections with the two stringers, so that the effective length of the angles free to buckle is not greater than 54 in. The details of these intersections are shown on Plate II.*a*.

The stresses in the two systems of triangulation are taken as equally divided between them. The stress in each diagonal in the first panel from the centre for the fixed and moving load is—

$$\pm\left\{\frac{6{\cdot}25}{6}(1 + 2 + 3)1{\cdot}8 + 2{\cdot}7\right\} = \pm\, 13{\cdot}95, \text{ or } \pm\, 14 \text{ units (nearly).}$$

In a similar manner the stresses in the second panel from the centre and the end panel are found—

$$\pm\,\frac{6{\cdot}25}{6}\,(1 + 2 + 3 + 4)1{\cdot}8 \pm 8{\cdot}1 = \pm\, 26{\cdot}85, \text{ or } \pm\, 27 \text{ units.}$$

$$\pm\,\frac{6{\cdot}25}{6}(1 + 2 + 3 + 4 + 5)1{\cdot}8 \pm 13{\cdot}5 = \pm\, 41{\cdot}63, \text{ or } \pm\, 42 \quad ,,$$

The sizes of the angles provided in the three panels on each side of the centre are—

one angle $5 \times 3\frac{1}{2} \times \frac{3}{8}$; gross area 3·3 sq. in.; net area 1·6 sq. in.
one angle $5 \times 3\frac{1}{2} \times \frac{7}{16}$; gross area 3·53 sq. in.; net area 1·8 sq. in.
two angles $5 \times 3\frac{1}{2} \times \frac{3}{8}$; gross area 6·67 sq. in.; net area 3·2 sq. in.

The areas required in tension for a unit stress of 18 units are 0·8, 1·5 and 2·4 sq. in. respectively. The areas necessary for compression since the least radius of gyration of the two angles is 0·96 in., and of one angle 0·76 in., using Cooper's formula, $13{,}000 - 60\frac{l}{r}$, are 1·6, 3·1 and 4·3 sq. in. respectively.

Sway and Portal Bracing.—The sway bracing must be designed to resist the maximum wind pressure acting on each panel of the main trusses, and the portal bracing must be designed to resist the reactions at the ends of the top chords for the maximum wind pressure. The method adopted of designing this bracing is illustrated in Plates II. and II.*a*.

Sway Bracing (Fig. 388).—The maximum wind pressures acting on the top panel points m and n are 3 units, producing equal and opposite reactions at the points d and d'. The decrease in the vertical reaction at d and the increase at d' is—

$$\frac{6 \times 28 \times 12}{194} = 10{\cdot}4 \text{ units.}$$

The bending moment at w' is—

$$-3 \times 28 + 10{\cdot}4 \times 11 = 30{\cdot}4.$$

The stress in $w'z'$ is found by taking moments about n—

$$\therefore 4 \text{ times the stress in } w'z' = -\,3 \times 28 + \frac{10{\cdot}4 \times 194}{12} = 84{\cdot}1,$$

$$\therefore \text{stress in } w'z' = \frac{84{\cdot}1}{4} = 21 \text{ units.}$$

The stress in $wz = -21$ units.

The tension in wm and the compression in $w'n$ are the horizontal components of the stresses in *wz* and *w'z'* respectively or about 15 units. The bending moment in *mn* increases uniformly from zero at the centre to its maximum value at *w* or *w'*, where it is 30·4 units, and then decreases uniformly towards *m* or *n*, where it is zero. The shear in *mn* is 10·4 units between *w* and *w'*, and between *w'* and *n* and *w* and *m* it is about 4·7 units. The sections and areas provided in the design, Plate II.*a*, are—

For the knee braces *wz* and *w'z'*, two angles 3½ in. × 3½ in. × ½ in., giving a gross area of 6·5, and a net area of 2·5 sq. in.; the value of *r* is 1·08, and the ratio $\frac{l}{r} = 48$. Cooper's formula, $13{,}000 - 60\frac{l}{r} =$ 10,120 pounds = say 10 units. The minimum areas required are therefore 2·1 and 1·2 sq. in. respectively for the compressive and tensile stresses of 21 units.

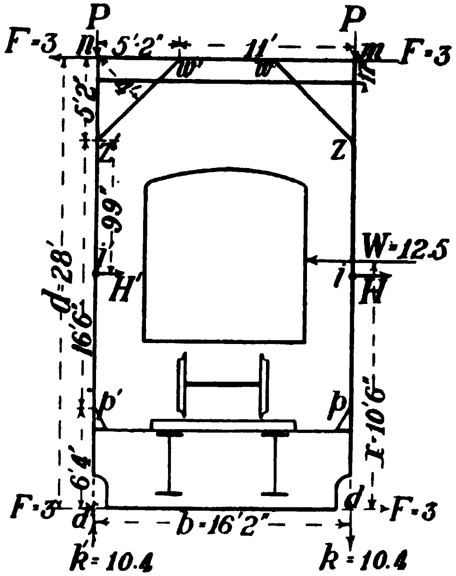


FIG. 388.

The horizontal girder *mn* has a total depth of 17 in. and an effective depth of 15 in. The four angles at the top and bottom are each 3 in. × 2½ in. × ⅜ in., giving a gross area in the compression flange of 3·8 sq. in. and a net area in tension of 1·6 sq. in. The area required in the top flange is $\frac{30{\cdot}4 \times 12}{15 \times 17} = 1{\cdot}43$ sq. in. due to the bending moment, and also an amount necessary to resist a total load of 3 units acting in the centre line of the column *mn* and equally distributed over the four angles. This will not be greater than 0·3 sq. in., so that the total area required is 1·73 sq. in., whereas 3·8 sq. in. are provided.

Portal Bracing (Fig. 389).—The wind loads acting at the points *o* and *o'* are each 7·5 units, and the horizontal reactions may be assumed equal to 7·5 units each. The lower ends of each of the inclined members may be considered as fixed at the points *a* and *a'*. They may also be considered as fixed at the points *y* and *y'*. The members *oa* and *o'a'* are distorted by the wind loads, and there must be a point of inflection somewhere between yy' and aa'. It will be assumed to be midway between the centres of the end pins and the lowest points y and y' of the bracing. To find the vertical reactions R and R′ we must take moments about the point i', where the bending moment is zero—

$$\frac{2 \times 7{\cdot}5 \times 286{\cdot}5}{194} = 22{\cdot}15 \text{ units.}$$

The bending moment about the point x is zero, and there is no stress in uu, uo and $u'o'$. These bars shorten the length of the compression members xy and xy'. The stress in xy' is equal to $22{\cdot}15 \sec \theta = -30{\cdot}16$ units, and in $xy = 30{\cdot}16$ units. For the stress in ox, take moments about y, then—

$$7{\cdot}5\,(181{\cdot}5 + 105) + ox \times 105 = 0.$$

The stress in $ox = -20{\cdot}46$ units. In a similar manner the stress in $o'x$ may be found to be $20{\cdot}46$ units. The tension in yi and the compression $y'i'$ are each equal to $22{\cdot}15$ units. The portions oy and $o'y'$ are not subjected to any direct stress. If there is a cross-girder at the ends attached to the inclined compression members there would be no question about the ends a and a' being fixed. In the design, Plate II.*a*, the member uu' is continued to the end posts and forms with the members ou and $o'u'$ a braced strut 39 in. deep. The members xy and xy' are each formed with two angles, $5 \times 3\frac{1}{2} \times \frac{3}{8}$, giving a gross area of 6·67 sq. in. and a net area of 3·2 sq. in. The safe stress in compression, taking the length $u'y'$ as 72 in., and the least radius of gyration of the two angles as 0·96, the ratio $\frac{l}{r} = 75$, and—

$$13000 - 60\,\frac{l}{r} = 8500 \text{ pounds per sq. in.}$$

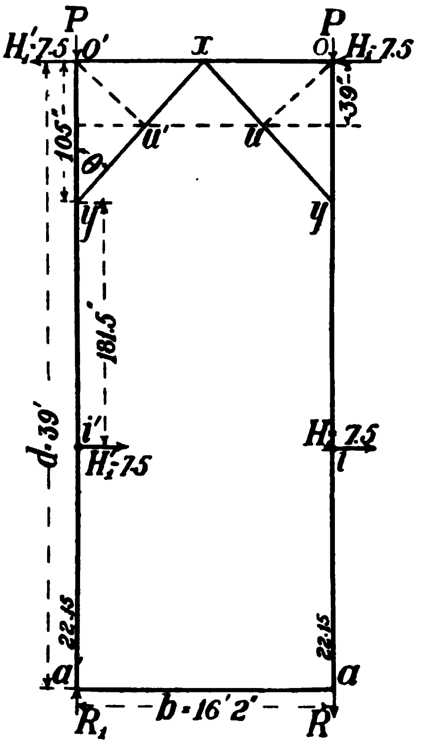


Fig. 389.

or 8·5 units. The area required is $\frac{30{\cdot}16}{8{\cdot}5} = 3{\cdot}5$ sq. in. The area required in tension is—

$$\frac{30{\cdot}16}{17} = 1{\cdot}8 \text{ sq. in.}$$

The points of inflection, i, i', were assumed to be midway between the levels yy' and aa', but with the form of bracing adopted it will be slightly lower.

The Overturning Effect of the Wind on the Train (Fig. 388).—The effect of the wind acting on the train is to increase the reactions of the floor beams on the leeward side and diminish it on the windward side, increasing the compression on the main compression members on the lee side and diminishing it on the wind side by an amount found in the following manner:—

$$\frac{12{\cdot}5 \times 10{\cdot}5 \times 12}{194} = 8{\cdot}12 \text{ units.}$$

The effect of the wind loads at the top panel points was found to be 10·4 units, but since this stress was due to 50 pounds wind pressure on the exposed area, only three-fifths could exist with the train on the bridge, or 6·24 units. The total increased stress on the main compression members is 8·12 + 6·24 = 14·36 units.

Reversal of Stress in the Lower Chords.—With a wind pressure of 50 pounds per sq. ft., and with no live load upon the bridge, the stresses in the bottom chords on the wind side may be reversed from tension to compression; and since the eyebars are only designed for tensile stress, it is necessary to introduce compression members riveted to the main posts designed to resist the difference between the dead load tensile stresses and the wind pressure compressive stresses. In the Pratt truss, shown in Plate II.*a*, there is no reversal of stress since in each panel the dead load tensile stress exceeds the wind load compression stress.

Stresses in Members of Main Trusses due to Bending.—The top and bottom members of a Pratt truss are subjected to bending under their own weight, and the stresses are not uniformly distributed over the cross-section. If M_1 denote the maximum bending moment producing a central deflection v_o, and P the total direct stress applied at the ends, whether tensile or compressive; also, if M denotes the resultant bending moment, then—

$$M = M_1 \pm Pv_o.$$

The upper sign is used when P produces compression, and the lower sign when it produces tension in the member. Let σ_1 denote the direct stress and a the area of the member, I the moment of inertia, r the radius of gyration, σ_2 the stress due to bending, l the length of the member, y the distance from the neutral axis to the extreme fibre where the stress is σ_2. Then it has been shown in Chapter X. that—

$$v_o = \frac{5}{48}\frac{\sigma_2 l^2}{\mathrm{E}y}.$$

In the following approximate method we may write with sufficient accuracy—

$$v_o = \frac{\sigma_2 l^2}{10\mathrm{E}y}, \ \mathrm{M} = \frac{\sigma_2 \mathrm{I}}{y}$$

$$\therefore \ \mathrm{M} = \mathrm{M}_1 \pm \frac{\mathrm{P}\sigma_2 l^2}{10\mathrm{E}y} \quad \therefore \ \sigma_2 = \frac{\mathrm{M}_1 y}{\mathrm{I} \mp \frac{\mathrm{P}l^2}{10\mathrm{E}}}$$

$$\text{or } \mathrm{M} = \frac{\mathrm{M}_1}{1 \mp \frac{\sigma_1 l^2}{10\mathrm{E}r^2}}.$$

The main vertical posts of a Pratt truss are subjected to bending under wind pressure and the inclined end members from the portal bracing. These members may be considered as subjected to a direct compressive stress and also to a bending moment M. Two cases occur—(1) when the floor beams are not riveted to the main vertical post, and (2) when they are riveted, as in Plate II.*a*.

In the first case the vertical posts may be considered as hinged at the base, and the bending moment at z' on the leeward side is $\mathrm{M}_1 = \mathrm{H}l$, where H is the horizontal reaction at the foot of the post and l the distance $d'z'$. The deflection at d' will be (Fig. 388)—

$$v_o = \frac{\sigma_2 l^2}{3\mathrm{E}y} \text{ and } \mathrm{M} = \frac{\mathrm{M}_1}{1 - \frac{\sigma_1 l^2}{3\mathrm{E}r^2}}$$

$$\text{where } \sigma_1 = \frac{\mathrm{P} + \frac{\mathrm{F}d}{b}}{\mathrm{A}}.$$

In the second case the floor beams may be considered as fixing the ends of the main vertical posts in a direction across the bridge in regard to bending in this direction, so that there is a point of inflection i' situated between p' and z'. The horizontal reactions H may be assumed to act at i' and i midway between $p'\ p$ and $z'\ z$. The transferred load due to the wind acting on the train may be assumed to act vertically on the main post with an eccentricity ϵ, equal to about half the depth of the cross-section of the post. The length of $i'\ z'$, Fig. 388, may be denoted by l', then—

$$v_o = \frac{\sigma_2 l'^2}{3\mathrm{E}y}$$

$$\mathrm{M}_1 = \mathrm{H}l' - \frac{\mathrm{W}x}{b}\epsilon$$

where $x = id$.

$$\text{and } M = \frac{M_1}{1 - \frac{\sigma_1 l^2}{3Er^2}} \text{ where } \sigma_1 = \frac{P + \frac{Wx\epsilon}{b}}{A}.$$

In the inclined members, Fig. 389, the moments at y and y_1 are due to the maximum horizontal reactions from wind acting at the points o and o', and these are assumed to act at the points of inflection i and i'. The transferred load producing an increased compression on the leeward post is $\frac{2H'_1 \times o'\,i'}{b}$. The total compression in the leeward post is $P + \frac{2H'_1 \times o'\,i'}{b} = P_1$; $\sigma_1 = \frac{P_1}{A}$. The moment at y' is $H_1' \times i'\,y' = M_1$. $\sigma_2 = \frac{M_1 c}{I - \frac{P_1 l^2}{10E}}$ where c = the distance from the neutral axis to the extreme fibre and l the length $i'\,y'$.

A few examples will now be given in order to illustrate the application of the foregoing approximate method in connection with the Pratt truss through bridge illustrated in Plates II., II.*a*. Eye-bars: *cd*, Plate II., consists of four bars, each $6 \times 1\frac{3}{8}$ in., giving a total area of 33 sq. in. The equivalent dead load direct stress is 556 units, or 139 units on each of the four bars. The weight of the bar is 33·15 pounds per ft. $I = 24{\cdot}7$, $y = 3$, $l = 300$ in., $E = 30{,}000$ units; then—

$$\sigma_1 = \frac{556}{33} = 16{\cdot}848 \text{ units}; \quad M_1 = \frac{Wl}{8} = \frac{25 \times 33{\cdot}15 \times 300}{8} = 31078$$

inch pounds.

$$\therefore M_1 y = 93234, \quad \frac{Pl^2}{10E} = 41{\cdot}7$$

$$\therefore \sigma_2 = \frac{M_1 y}{I + \frac{Pl^2}{10E}} = 1404 \text{ pounds per sq. in.}$$

$$\therefore \sigma = \sigma_1 + \sigma_2 = 18{\cdot}252 \text{ units per sq. in.}$$

Chord member: *mn*, Plate II., consists of plates and channel bars, having a total area of 44 sq. in. The equivalent dead load is 628·7 units. $I = 1976$, $y = 8$, $l = 300$, $W = 25 \times 150 = 3750$ pounds.

$$M_1 = \frac{3750 \times 300}{8} = 140625 \text{ inch pounds}, \quad M_1 y = 1125000$$

$$\frac{Pl^2}{10E} = \frac{628{\cdot}7 \times 90000}{300000} = 188{\cdot}61.$$

$$\therefore \sigma_2 = \frac{M_1 y}{I - \frac{Pl^2}{10E}} = 629 \text{ pounds per sq. in.}$$

$$\sigma_1 = \frac{628{\cdot}7}{44} = 14{\cdot}272 \text{ units.} \quad \therefore \sigma = \sigma_1 + \sigma_2 = 14{\cdot}9 \text{ units per sq. in.}$$

Vertical post: *nc*, Plate II. The equivalent dead load stress is 155·3 units; the distance $i'z'$, Fig. 388, is 99 in.; the area of the cross-section is 14·8 sq. in. $I = 288$, $r = 4{\cdot}43$; also the eccentricity $\epsilon = 6$ in. The wind loads at *m* and *n*, when the train is on the bridge, are 1·8 units, and the load transferred by the wind on the train to the leeward side is 8·12 units. $P = 155{\cdot}3 + 8{\cdot}12 = 163{\cdot}42$ units—say 164 units.

$$M_1 = 1{\cdot}8 \times 99 - 8{\cdot}12 \times 6 = 178{\cdot}2 - 48{\cdot}2 = 129{\cdot}48\text{—say } 130 \text{ units.}$$

$$M_1y = 130 \times 6 = 780;$$

$$\frac{Pl^2}{3E} = \frac{164 \times 9801}{90000} = 18 \text{ nearly.}$$

$$\therefore \sigma_2 = \frac{780}{288 - 18} = 2{\cdot}9 \text{ units nearly.}$$

$$\sigma_1 = \frac{164}{14{\cdot}8} = 11{\cdot}08 \text{ units.}$$

$$\therefore \sigma = \sigma_1 + \sigma_2 = 13{\cdot}98 \text{ units—say 14 units.}$$

Cooper's formula gives the allowed stress—

$17000 - 90 \times 22{\cdot}3 = 14993$ pounds per sq. in., or 14·993 units.

End post: *ao*, Plate II. The equivalent dead load stress is 539·75 units; the distance $i'y'$, Fig. 389, is 181·5 in.; the area of the cross-section is 46·9 sq. in. $I = 2100$, $r = 6{\cdot}7$, $c = 8$. The horizontal reactions at *i* and *i'* for a wind load of 30 pounds per sq. ft. is 4·5 units, and the moment at y'—

$$M_1 = 4{\cdot}5 \times 181{\cdot}5 = 816{\cdot}75, \text{ or } 816750 \text{ inch pounds.}$$

$$M_1c = 6534; \quad \frac{Pl^2}{3E} = \frac{553 \times 32943}{90000} = 202{\cdot}4,$$

where $P = 539{\cdot}75 +$ the transferred load $= 539{\cdot}75 + 13{\cdot}29 = 553$.

$$\sigma_2 = \frac{6534}{2100 - 202{\cdot}4} = 3{\cdot}5 \text{ units.}$$

$$\sigma_1 = \frac{553}{46{\cdot}9} = 11{\cdot}8 \text{ units.}$$

$$\sigma = \sigma_1 + \sigma_2 = 15{\cdot}8 \text{ units.}$$

Cooper's formula gives $\sigma = 15{\cdot}3$ units.

BRIDGE OVER HUNTER RIVER AT LUSKINTYRE, N.S. WALES.

This bridge was designed by the engineers of the Public Works Department, N.S. Wales, and is an excellent example of a steel Pratt truss highway bridge with riveted connections. The main details of this bridge are illustrated in Plate II.*b*, from which it will be seen that the span of the main trusses is 198 ft., centres of bearings consisting of eleven panels each 18 ft. long and an effective depth of 25 ft. The two main trusses are spaced 22 ft. 2 in. centre to centre, and the steel cross-girders spaced 18 ft. apart centre to centre have an effective depth of 24 in. The longitudinal stringers are of ironbark timber 12 × 12 in. in cross-section and spaced 44 in. apart centre to centre across the bridge. A continuous floor of hardwood timber is laid over the stringers, 4 in. thick, forming a roadway 18 ft. wide between kerbs, and one footway is provided for, to be carried on steel cantilevers outside one truss, but this has been omitted for the present, and is not shown on Plate II.*b*

Loading and Unit Stresses.—The bridge has been designed for a live load of 84 pounds per sq. ft. on the roadway and footway, and a traction engine weighing 16 tons, shown in Fig. 390.

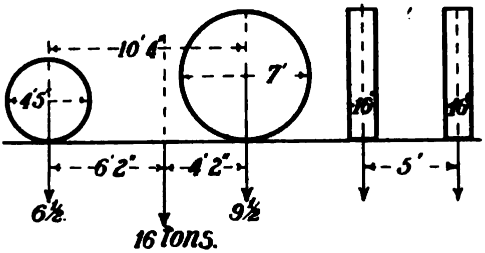


FIG. 390.

The impact effect is provided for by Pritchard's formula—

$$I = cl\left(\frac{l}{l + d}\right),$$

where c is taken at 0·15 for the live load of 84 pounds per sq. ft. and 0·30 for the traction engine.

Tension. The unit stress for dead load, live load and impact = 17,000 lbs. per sq. in.

The unit stress for wind = 20,000 lbs. per sq. in.

Compression. The unit stress for dead load, live load and impact

$$= 17{,}000 - 80\frac{l}{r} \text{ lbs. per sq. in.}$$

The unit stress for wind = $20{,}000 - 100\frac{l}{r}$ lbs. per sq. in.

Alternating Stresses. The unit stress in tension

$$= 15{,}000\left(1 - \frac{1}{2}\frac{\text{min}}{\text{max}}\right) \text{ lbs. per sq. in.}$$

The unit stress in compression $= \left(15{,}000 - 60\frac{l}{r}\right)\left(1 - \frac{1}{2}\frac{\text{min}}{\text{max}}\right)$ lbs. per sq. in.

Shear. The unit stress for steel = 12,000 lbs. per sq. in.

The unit stress for wrought-iron = 9000 lbs. per sq. in.

Bearing Pressure. The unit stress for steel = 24,000 lbs. per sq. in.

The unit stress for wrought-iron = 16,000 lbs. per sq. in.

Pressure on Masonry. Basalt bed-stones = 500 lbs. per sq. in.

Concrete = 200 lbs per sq. in.

Modulus of Rupture of Ironbark = 6 tons per sq. in.

Pressure on Expansion Rollers = $300d$, but not to exceed 3000 lbs. per lineal in.

The calculations will not be given in the same detail as in the case of the Pratt truss bridge, Plates II. and II.*a*, as this would mean repetition.

Stringers. 12 × 12 in. in cross-section, 18 ft. span, spaced 40 in. centres across the bridge.

Dead load. The weight of the timber is 75 pounds per cub. ft. and on one stringer is 3000 pounds.

The bending moment due to the dead load is—

$$\frac{3000 \times 18 \times 12}{8 \times 2240} = 36{\cdot}2 \text{ inch tons.}$$

The bending moment due to the traction engine bearing equally on two stringers is, since the greatest value occurs with the heaviest pair of wheels in the centre of the span—

$$\frac{4{\cdot}75 \times 18 \times 12}{4} \times \frac{36}{44} = 210{\cdot}6 \text{ inch tons.}$$

Impact effect, since $\dfrac{l}{l + d} = \dfrac{210{\cdot}6}{36{\cdot}2 + 210{\cdot}6} = 0{\cdot}86.$

$$\therefore \cdot 30 \times 0{\cdot}86 \times 210{\cdot}6 = 54{\cdot}8 \text{ inch tons.}$$

The total bending moment is—

$$36{\cdot}2 + 210{\cdot}6 + 54{\cdot}8 = 302 \text{ inch tons.}$$

This is assumed to be equally distributed over each of the six beams, giving a moment of resistance—

$$\frac{\cdot 6 \times 12 \times 12 \times 12}{6} = 1728 \text{ inch tons.}$$

The factor of safety $= \dfrac{1728}{302} = 5{\cdot}7$ on this assumption.

Cross-girders spaced 18 ft. centres, 21 ft. span, 24 in. effective depth.

Dead load, including handrail, posts, kerbs, floor and stringers = 9·5 tons on one cross-girder.

The bending moment due to the dead load is—

$$\frac{9{\cdot}5 \times 21 \times 12}{8} = 299{\cdot}3 \text{ inch tons.}$$

The maximum bending moment due to the traction engine which loads the girder at the points of attachment of the four central stringers is—

$$6{\cdot}12 \times 126 - 1{\cdot}1 \times 66 - 5{\cdot}02 \times 22 = 588{\cdot}02 \text{ inch tons.}$$

Impact effect, since $\frac{l}{l + d} = \frac{588\cdot02}{887\cdot38} = 0\cdot61.$

$$\therefore 0\cdot30 \times 0\cdot61 \times 588\cdot02 = 117\cdot62 \text{ inch tons.}$$

The total bending moment is—

$$299\cdot3 + 588\cdot02 + 117\cdot62 = 1005 \text{ inch tons.}$$
$$= 2{,}251{,}200 \text{ inch pounds.}$$

The area required is—

$$\frac{2{,}251{,}200}{24} \times 17{,}000 = 5\cdot51 \text{ sq. in.}$$

The area provided by two angles 4 × 4 × $\frac{1}{2}$ in. less two holes for $\frac{7}{8}$ rivets is 6·5 sq. in.

Main Trusses.—The total dead load on one truss was found to be 1400 pounds per lineal ft., and the live load for one footway and half the roadway is—

$$(9\cdot0 + 5\cdot0)84 = 1176 \text{ pounds per lineal foot.}$$

The panel point loads will be assumed to be concentrated at the points L_1 to L_{10} (Fig. 391), and are—

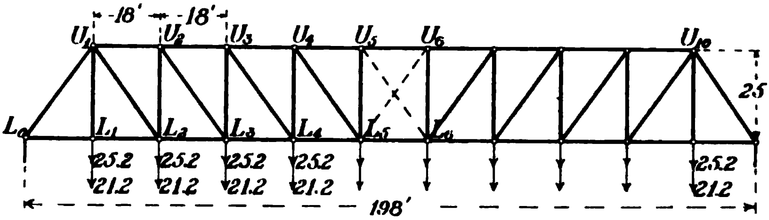


FIG. 391.

Dead load = 1400 × 18 = 25,200 pounds = say 25·2 units of 1000 pounds.

Live load = 1176 × 18 = 21,168 pounds = say 21·2 units of 1000 pounds.

$$\text{Tan } \theta = \frac{18}{25} = \cdot72\,;\ \sec \theta = \frac{30\cdot806}{25} = 1\cdot232.$$

$25\cdot2 \sec \theta = 31$ units, $21\cdot2 \sec \theta = 26$ units.

$25\cdot2 \tan \theta = 18\cdot15$ units, $21\cdot2 \tan \theta = 15\cdot2$ units.

The stresses in the top and bottom members due to these panel loads are given in the following table in units of 1000 pounds :—

Table LXXII.

Member.	Dead load stress.	Live load stress, roadway and footway.	Impact $\cdot15l\left(\frac{l}{l+d}\right) = \cdot07l$.	Total stress.
L_0U_1	− 155	− 130	− 9	− 294
U_1U_2	− 163	− 137	− 10	− 310
U_2U_3	− 218	− 183	− 13	− 414
U_3U_4	− 254	− 213	− 15	− 482
U_4U_5	− 272	− 228	− 16	− 516
U_5U_6	− 272	− 228	− 16	− 516
$L_0L_1 = L_1L_2$	91	76	5	172
L_2L_3	163	137	10	310
L_3L_4	218	183	13	414
L_4L_5	254	213	15	482
L_5L_6	272	228	16	516

The stresses in the vertical and diagonal members are given in the following table in units of 1000 pounds :—

Table LXXIII.

Member.	Dead load stress.	Live load stress, roadway and footway.	Impact ratio $\cdot15l\left(\frac{l}{l+d}\right)$	Addition for impact.	Total stress.
U_1L_1	− 25·2	21·2	·07*l*	1·48	47·9
U_2L_2	− 75·6	− 68·3	·07*l*	− 4·85	− 148·9
U_3L_3	− 50·4	− 53·9	·08*l*	− 4·32	− 108·6
U_4L_4	− 25·2	$\begin{Bmatrix}-40{\cdot}4\\+19{\cdot}3\end{Bmatrix}$	·09*l*	− 3·64	− 69·2
U_5L_5	0	$\begin{Bmatrix}-28{\cdot}9\\+28{\cdot}9\end{Bmatrix}$	Alternating	No impact	∓ 28·9
U_1L_2	124·2	106·7	·075*l*	8·00	238·9
U_2L_3	93·2	85·3	·072*l*	6·14	184·6
U_3L_4	62·1	66·4	·078*l*	5·18	133·7
U_4L_5	31·1	$\begin{Bmatrix}49{\cdot}8\\-23{\cdot}7\end{Bmatrix}$	·092*l*	4·63	85·5
U_5L_6	0	35·6	·15*l*	5·34	40·9

It may be shown that the effect of the traction engine on the roadway combined with the live load of 84 pounds per sq. ft. on the footway is less than the stresses given in Table LXXIII. for every member of the truss excepting U_1L_1. It has been assumed that the whole panel point load of 25·2 units is concentrated at the bottom joints of the truss, whereas it would have been more accurate to have taken 8·4 at the top joints and 16·8 at the bottom joints. The maximum reaction at U_1L_1 for the traction engine in the worst position may be shown to be 21·4 units plus the impact effect equal to 3·6 units.

The live load on the footway produces a stress in U_1L_1 of 7·6 units plus the impact addition 0·3 units, giving 7·9 units. The stress in U_1L_1 is therefore—

$$16{\cdot}8 + 21{\cdot}4 + 3{\cdot}6 + 7{\cdot}9 = 49{\cdot}7 \text{ units,}$$

instead of 47·9 units as given in Table LXXIII.

The stresses due to wind have been sufficiently illustrated in the Pratt truss railway bridge, Plates II. and II.*a*, and they increase the stresses in the upper chords and sloping ends; they do not, however, reverse the stresses in the lower chords even when a wind pressure of 50 pounds per sq. ft. on the trusses is combined with the effect of the dead load. These stresses are not given in detail as in the case of the Pratt truss railway bridge, but a summary of the total stresses, the areas required in sq. in., also the actual sections and areas provided are given in Table LXXIV. The details of the various members and the riveted connections are sufficiently illustrated in Plate II.*b*, and the portions omitted in the calculations can be supplied by the reader after carefully studying the Pratt truss railway bridge.

TABLE LXXIV.

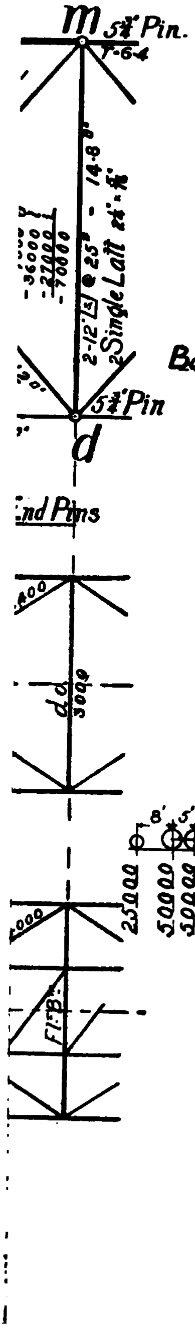
SUMMARY OF STRESSES AND AREAS OF SECTIONS.

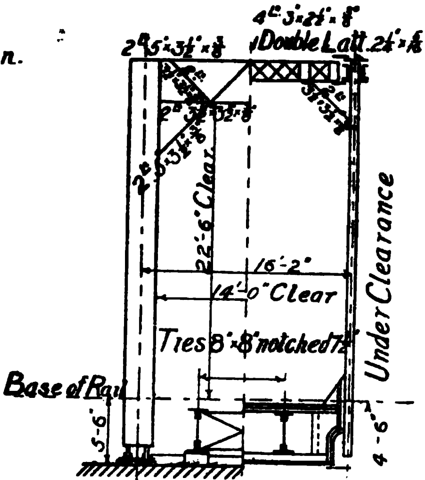
Member of truss.	Units of 1000 pounds.		Area required in sq. in.	Sections and areas provided, sq. in.
	Max. stress.	Unit stress.		
L_0U_1	- 406	14	29·0	22 × $\frac{3}{8}$ = 8·25 4Ls, 3 × 3 × $\frac{1}{2}$ = 11·00 2 × 15 × $\frac{3}{8}$ = 11·25 } 30·5
U_1U_2	- 310	14	22·1	. . . . . 30·5
U_2U_3	- 413	14	29·5	. . . . . 30·5
U_3U_4	- 482	14	34·5	As shown above = 30·5 2 × 8$\frac{3}{4}$ × $\frac{3}{8}$ = 6·5 } 37·0
U_4U_5	- 517	14	36·9	. . . . . 37·0
U_5U_6	- 517	14	36·9	. . . . . 37 0
$U_0L_1 = L_1L_2$	172	17	10·1	4Ls, 4 × 4 × $\frac{1}{2}$ = 13·28 2, 15 × $\frac{3}{8}$ = 10 } 23·3 net.
L_2L_3	310	17	18·2	. . . . . 23·3 ,,
L_3L_4	414	17	24·3	4Ls, 4 × 4 × $\frac{1}{2}$ = 13·28 2, 15 × $\frac{3}{8}$ = 10·00 2, 7 × $\frac{1}{2}$ = 5·00 } 28·3 ,,
L_4L_5	482	17	28·4	28·3 ,,
L_5L_6	516	17	30·4	Same as above = 28·3 2, 13$\frac{1}{2}$ × $\frac{1}{4}$ = 5 } 33·3 ,,
U_1L_1	50	17	3·0	2, 10 × 3*c*, 22$\frac{1}{2}$ lb. . 11·5 ,,
U_2L_2	- 149	14	13·1	,, gross 13·2
U_3L_3	- 109	14	9·5	,, . . 13·2
U_4L_4	- 69	14	6·1	,, . . 13·2
U_5L_5	∓ 29	- 5·4	5·3	,, . . 13·2
		+ 7·5	3·9	,, . . 11·5
U_1L_2	239	17	14·0	2, 10 × $\frac{7}{8}$ . . 15·5 net.
U_2L_3	185	17	10·9	2, 10 × $\frac{5}{8}$ . . 11·3 ,,
U_3L_4	134	17	7·9	2, 8 × $\frac{5}{8}$ . . . 8·7 ,,
U_4L_5	86	17	5·0	2, 7 × $\frac{1}{2}$ . . . 6·1 ,,
U_5L_6	41	17	2·4	2, 6 × $\frac{3}{8}$ . . . 3·8 ,,

TRUSS. PLATE II.

- 10.5
- 26.5
- 7.0

44.0

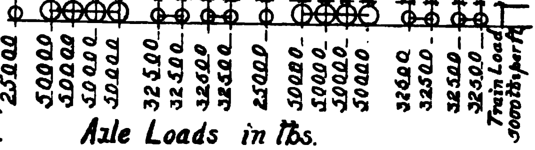


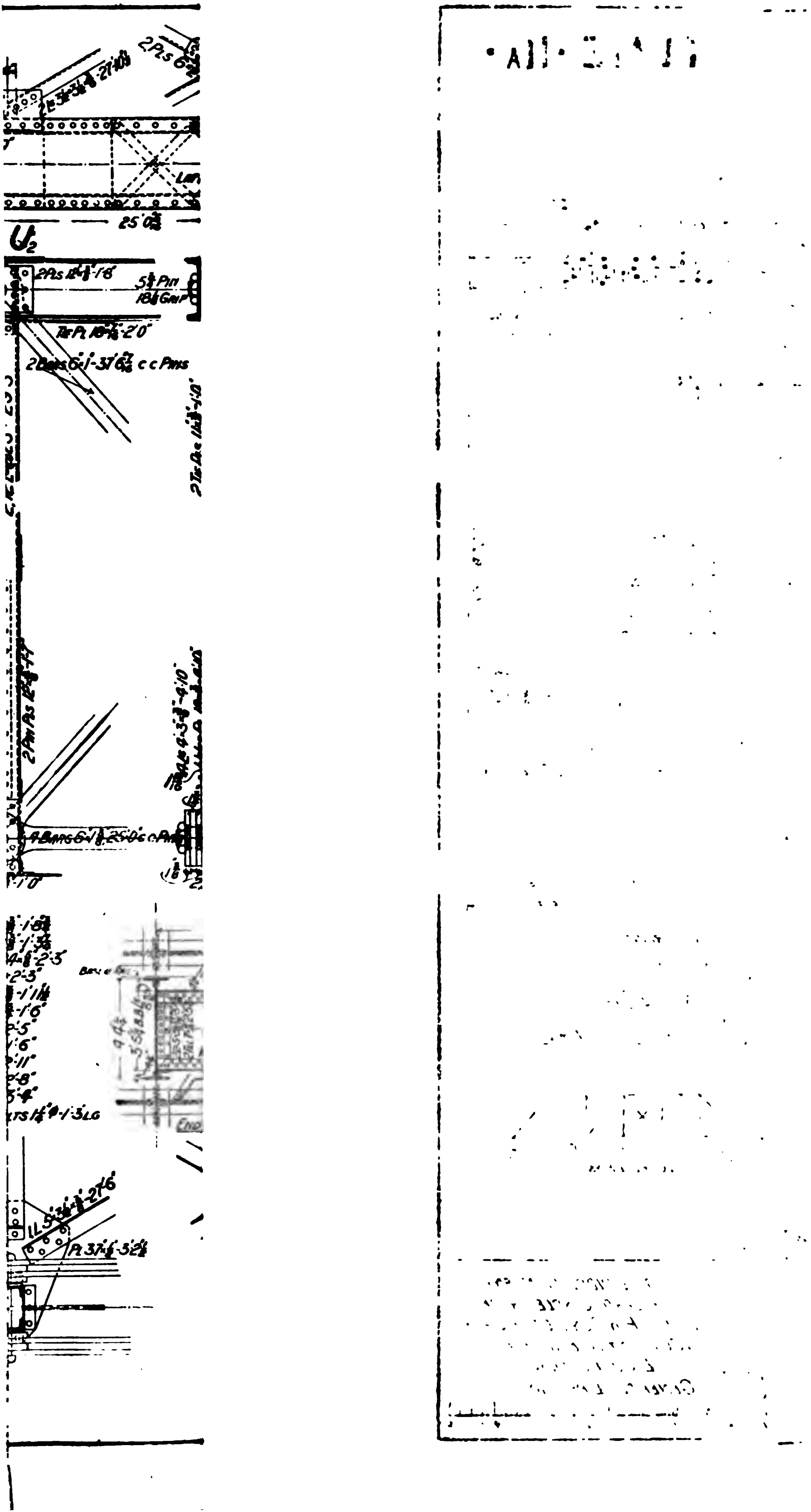


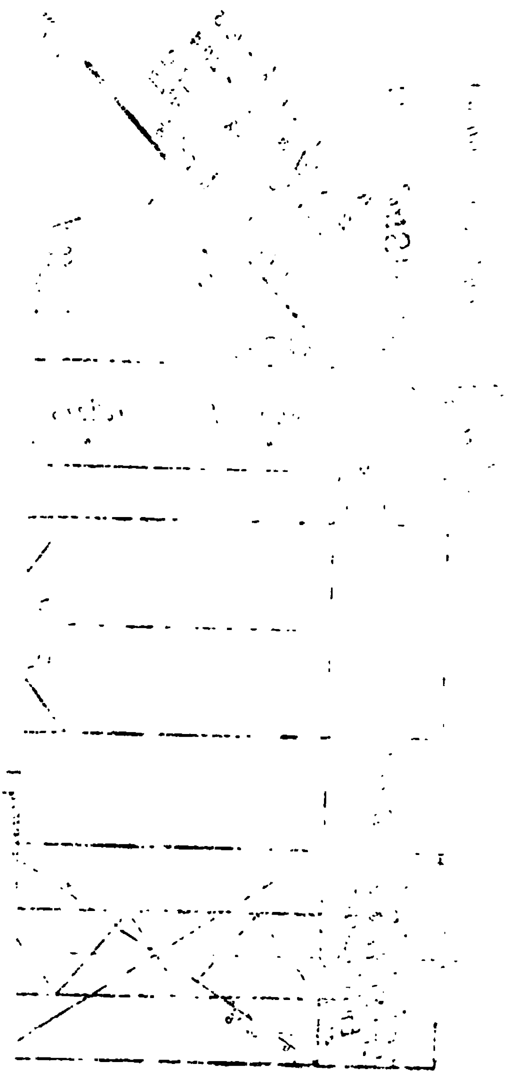
Half End View Half Int. Sect

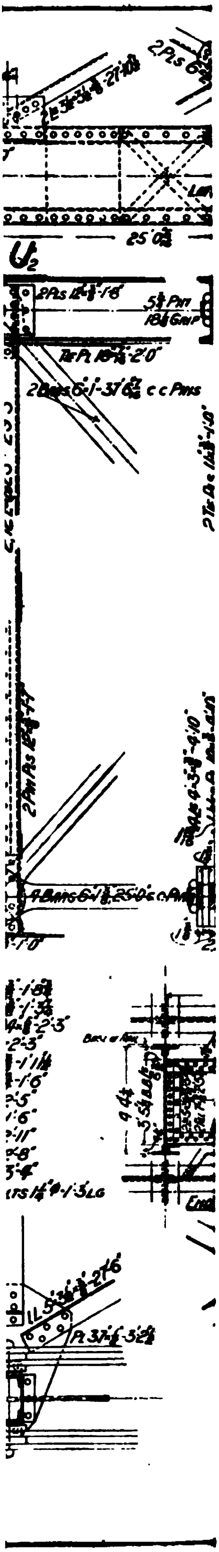
Cooper's E_{50} Engine.

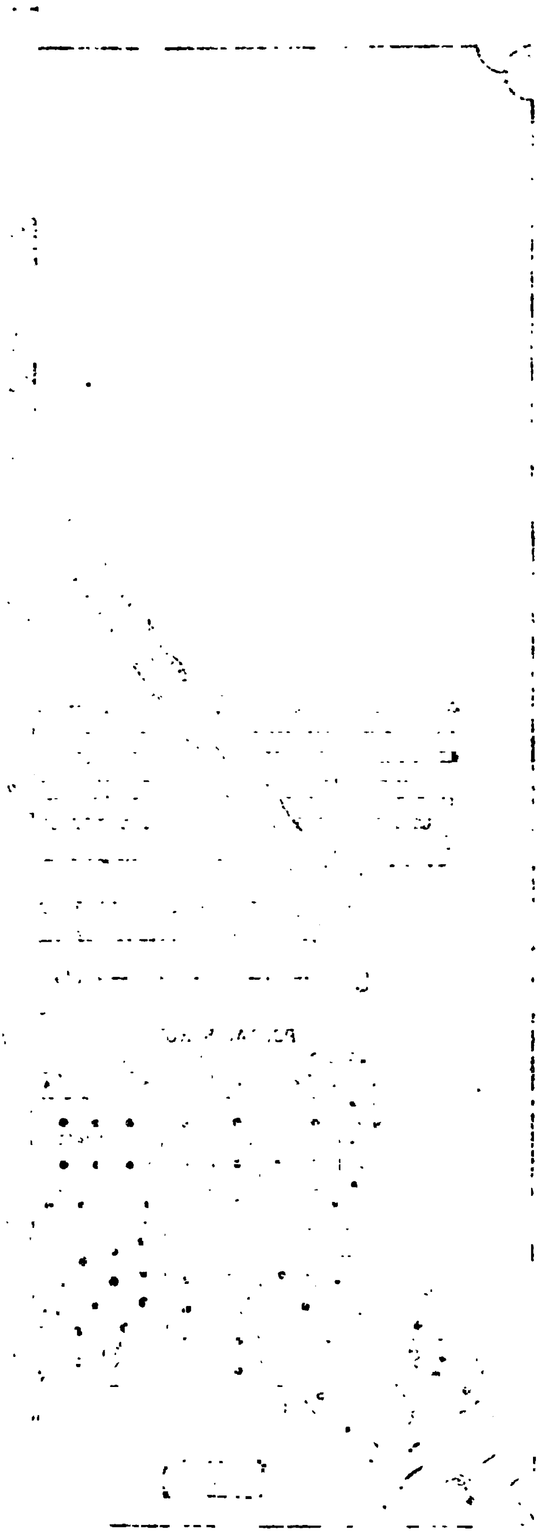
Load Diagram.




2 Pls 6
25' 0 3/8"
U2
2 Pls 18 x 5/8 x 1'6"
5 1/2 Pin
18 5/8 Grip
Tie Pl 18 x 5/8 x 2'0"
2 Bars 6 x 1 - 37' 6 27/32 c c Pins
2 Pin Pls 12 x 5/8 x 7'7"
4 Bars 6 x 1 1/8 25'0" c c Pins
1'0"
Rivets 1 1/4 φ - 1'3 Lg
End
1 L 5 x 3 1/2 x 3/8 27'6"
Pl 37 x 3/8 x 3'2 1/2"




2 Pls 6
2L 3½x3½x⅜-21'10⅞
25' 0⅜"
U2
2 Pls 12x⅜-1'6"
5¼ Pin
18⅜ Grip
Tie Pl 18x⅝-2'0"
2 Bars 6x1-37'6 7/16 c c Pins
2 Tie Pls 18⅜-1'0"
2 Pin Pls 18x⅝-4'7"
4 Bars 6x1⅛-25'0 c c Pins
Rivets 1¼"⌀-1'3 Lg
Pl 37x⅝-5'2¼
1L 5x3½x⅜-27'6
Brg Bar
9'4¼
Lat



CHAPTER XIX.

DESIGN OF A PARKER TRUSS BRIDGE FOR A SINGLE LINE OF RAILWAY.

THIS bridge is illustrated in Plates III. and III.*a*, showing the stress sheet and the design of the various details. This bridge is the standard type adopted by the American Bridge Company, and has been designed from their own specifications. The calculation of the stresses in the main trusses will be considered sufficiently to illustrate the method as applied to trusses in which the chords are not parallel.

The data is as follows :—

Span of main trusses . . . . . . = 200 ft.
Number of panels . . . . = 8 ,,
Length of each panel . . . . . . = 25 ,,
Height of vertical members in the first bay U_1L_1 = 28 ,,
,, ,, ,, second bay U_2L_2 = 33 ,,
,, ,, ,, third bay U_3L_3 = 36 ,,
,, ,, ,, centre U_4L_4 = 36 ,,

The outline diagram of this truss is shown in Fig. 392, and the various stresses are written on Plate III. The calculations in regard to the longitudinal girders and the floor girders are also given on Plate III., so that only the stresses in the main trusses remain to be considered. The method of lettering the upper and lower joints is convenient and frequently adopted.

Dead Load.—The dead load at each panel point may be determined in the same manner as in the case of the Pratt truss. The weight of the two trusses was known to be 1520 pounds per lineal ft. See also Chapter XVI.

The panel point loads may be found in the following manner :—

Rails, sleepers and fastenings = 400 pounds per ft. = 5,000 at panel.
Floor including longitudinal and cross-girders, bracing, etc. . = 520 ,, = 6,500 ,,
Trusses . . . . . = 1520 ,, = 19,000 ,,
Total . . . . = 30,500 ,,

Messrs. Waddell and Hendrick's formula (Chapter XVI.) gives

31362·5 pounds per lineal ft. In this example we shall adopt 30·5 units of 1000 pounds, and divide the panel load between the upper and lower joints as follows :—

U_1 to U_7 are considered loaded with 9·5 units each.
L_1 to L_7 with 21 units each.

Live Load.—The live load is taken as represented by two of Cooper's standard locomotives E_{50} followed by a train of 5000 pounds per lineal ft.

Dead Load Stresses.—The dead load stresses may be determined in the following manner. Member L_0U_1. Take moments about the point L_1, then—

$$+ 25R_1 + \text{stress in } U_1L_0 \times 18{\cdot}6 = 0,$$
$$R_1 = 30{\cdot}5 \times 3{\cdot}5 = 106{\cdot}75,$$
$$\therefore \text{Stress } L_0U_1 = -\frac{106{\cdot}75 \times 25}{18{\cdot}6} = -143 \text{ units.}$$

Member U_1L_1. The stress in this member is 21 units.

Members L_0L_1 and L_1L_2. The stress in these members are each equal to $R_1 \tan \theta$—

$$\frac{106{\cdot}75 \times 25}{28} = 95{\cdot}3 \text{ units.}$$

Member U_1U_2. Take moments about the intersection of U_1L_2 and L_1L_2, 50 ft. from L_0, also the perpendicular distance of L_2 on U_1U_2 is 32·5 ft.—

$$\text{The stress } U_1U_2 \times 32{\cdot}5 + 106{\cdot}75 \times 50 - 30{\cdot}5 \times 25 = 0$$
$$\therefore \text{The stress } U_1U_2 = -140{\cdot}77, \text{ say 141 units.}$$

Member U_1L_2. The moment point is at the intersection of U_1U_2 and L_1L_2, 115 ft. to the left of L_0, the lever arm of the stress in U_1L_2 is 123 ft.—

$$\text{The stress in } U_1L_2 \times 123 - 106{\cdot}75 \times 115 + 30{\cdot}5 \times 140 = 0$$
$$\therefore \text{The stress in } U_1L_2 = 65 \text{ units.}$$

Member U_2L_2. The moment point is the same as for U_1L_2, but the lever arm of the stress in U_2L_2 is 165 ft.—

$$\text{The stress in } U_2L_2 \times 165 - 106{\cdot}75 \times 115 + 30{\cdot}5 \times 140 + 21 \times 65 = 0$$
$$\therefore \text{The stress in } U_2L_2 = -27{\cdot}52, \text{ say 28 units.}$$

Member L_2L_3. The moment point is U_2, and the lever arm of the stress in L_2L_3 is 33 ft.—

$$\text{The stress in } L_2L_3 \times 33 + 106{\cdot}75 \times 50 - 30{\cdot}5 \times 25 = 0$$
$$\therefore \text{The stress in } L_2L_3 = 138{\cdot}6, \text{ say 139 units.}$$

The remaining stresses may be found in a similar manner, and they are written on Plate III. thus :—

$U_2U_3 = -\{106{\cdot}75 \times 75 - 30{\cdot}5(25+50)\}\frac{1}{35{\cdot}8} \quad = -160$ units.

$U_3U_4 = -\{106{\cdot}75 \times 100 - 30{\cdot}5(25+50+75)\}\frac{1}{36} = -169$,,

$L_3L_4 = \quad \{106{\cdot}75 \times 75 - 30{\cdot}5(25+50)\}\frac{1}{36} \quad = \quad 159$,,

$U_2L_3 = \quad \{106{\cdot}75 \times 225 - 30{\cdot}5(250+275)\}\frac{1}{239} \quad = \quad 35$,,

$U_3L_4 =$ × The shear in the panel × sec a, since the chords are parallel, sec a = 1·22.

Shear = 106·75 − 91·5 = 15·25 units.

The stress in U_3L_4 = 15·25 × 1·22 = 19 units nearly.

$U_3L_3 = -\{106{\cdot}75 \times 225 - 30{\cdot}5(250+275) - 21 \times 300\}\frac{1}{300}$
$= -6$ units nearly.

If the counterbrace U_4L_3 acts, the stress in U_3L_3 is found by including the moment of U_4L_3 in the equation of moments. The lever arm of U_4L_3 is 246 ft.

$$\therefore \text{ The stress in } U_3L_3 = \frac{19 \times 246}{300} - 6 = 10 \text{ nearly.}$$

Live Load Stresses.—The maximum live load stresses in the various members of the truss, Fig. 392, may be found for any system of loading, such as Cooper's standard train loads, when the position of the loads has been arranged to produce the maximum stress in the member under consideration. In the case of the upper and lower members the position is determined in precisely the same manner as that illustrated in the case of the Pratt truss, Plates II. and II.*a*, using the criterion—

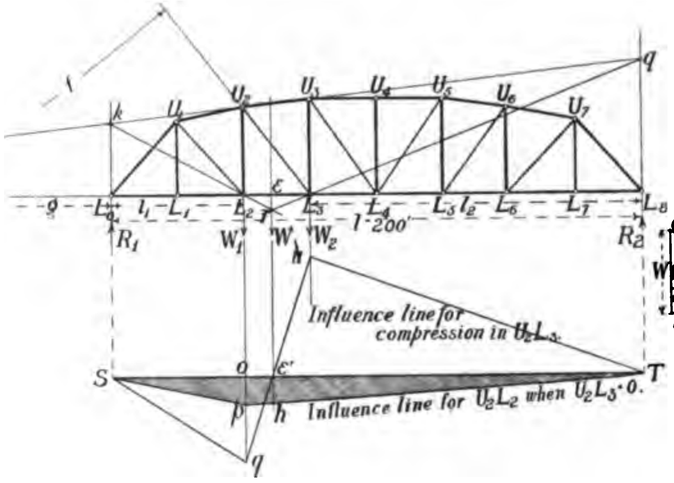

q
k
U1
U2
U3
U4
U5
U6
U7
ε
g
L0
l1
L1
L2
J
L3
L4
L5
l2
L6
L7
L8
1·200′
R1
W1
W
W2
R2
h
Influence line for compression in U2L3.
S
o
e′
T
p
h
Influence line for U2L2 when U2L3·0.
q

FIG. 392.

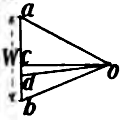


FIG. 393.

$$\frac{l'}{l} = \frac{P}{W}.$$

In the case of the diagonal and vertical members the criterion—

$$nP - W = a \text{ minimum}$$

does not apply.

Consider the stress in the member U_2L_3, and pass a section cutting the three members in the usual manner. Also produce U_3U_2 and L_3L_2 to intersect at the moment point for U_2L_3.

Let R_1 and R_2 denote the reactions at L_0 and L_8, then if there is no load at L_2, or to the left of L_2, the stress in U_2L_3, multiplied by its lever arm f, equals the reaction R_1 multiplied by its lever arm g—

$$\text{Stress in } U_2L_3 = \frac{R_1 g}{f}.$$

If there is a load at L_2, or to the left of this point, the stress in U_2L_3 is—

$$\frac{R'_1 g - W(g + l_1)}{f}.$$

$R'_1 - R_1$ is less than W, and $R_1 g$ is greater than $R'_1 g - W(g + l_1)$, hence the maximum stress in U_2L_3 occurs when there is no load at L_2, or to the left of this point. Again, a load at L_2 produces compression in U_2L_3, and a load at L_3 produces tension in U_2L_3, and therefore there must be some point between L_2 and L_3 where a load would not produce a stress in U_2L_3. Let ϵ denote this point and x the distance $L_3\epsilon$, also let np denote the span l, and l_1 and l_2 the distances L_0L_2 and L_3L_8 respectively. Then the moment of the stress in U_2L_3 about the intersection of U_3U_2 and L_3L_2 may be expressed thus—

$$\frac{wg}{2l}(l_2 + x)^2 - \frac{wx^2}{2p}(l_1 + g),$$

where w denotes a uniform load extending from L_8 to ϵ. For a maximum we have—

$$\frac{wg}{np}(l_2 + x) - \frac{wx}{p}(l_1 + g) = o$$

$$\therefore l_2 + x = nx\frac{l_1 + g}{g}.$$

If W denote the total load on the bridge, and P the load on the panel L_2L_3, not including the load at L_3, then the criterion for the position of any system of loading producing the maximum stress in U_2L_3 may be expressed thus—

$$W = nP\left(\frac{l_1 + g}{g}\right)$$

or as in the case of the Pratt truss we may write—

$$W - nP\left(\frac{l_1 + g}{g}\right) = \text{a minimum.}$$

In the Pratt truss, $g = \infty$, and $\frac{l_1 + g}{g} = 1$.

Since $\frac{l_1 + g}{g}$ is greater than unity, it follows there is a smaller load on the panel than in the Pratt truss when the diagonal receives its maximum stress. It can easily be shown that the distance $L_3\epsilon$ is—

$$x = \frac{l_2}{n\left(\frac{l_1 + g}{g}\right) - 1}$$

The point ϵ may be found also by the following graphical construction:—

Produce U_2U_3 to meet the directions of the reaction R_1 and R_2 at *k* and *q* respectively, Fig. 392, and draw lines through qL_3 and kL_2 to meet at *r*. A line drawn vertically through *r* will cut L_2L_3 at the point ϵ. This is shown by drawing *ab* to scale equal to W, and lines *ob* and *oa* parallel to *qr* and *kr*, then *kqr* is an equilibrium polygon, and *od* drawn parallel to *kq* gives *ad* and *db* the reactions at L_0 and L_8 respectively. The lines L_2r, rL_3 and L_2L_3 also form an equilibrium polygon, and *oc* drawn parallel to L_2L_3 gives *ac* and *cb* for the panel reactions W_1 and W_2 respectively. The line of action of the resultant of W_1 and R_1 must pass through the intersection of *qk* and L_3L_2, which is the centre of moments for U_2L_3. The moment of the resultant is therefore zero, and also the stress in U_2L_3. The stresses in the various members of the truss illustrated in Plate III. may now be completed.

Web Members.—The same diagram, Fig. 383, used to compute the live load stresses in the Pratt truss may also be used in the present case.

Member U_3L_4: Since the top members U_3U_4 and U_4U_5 are parallel to L_0L_8, the criterion for the position of the train load is exactly the same as for the Pratt truss. The wheel 3 is placed at the point L_4; the length of uniform load on the bridge is 4 ft.; the total load on the bridge is 365 units; and the load in front of L_4, not including that at L_4, is 37·5 units. $W - nP = 365 - 8 \times 37{\cdot}5 = 65$. If the loads are moved to the right or the left a small amount the difference will be greater, so that the position taken will give the greatest stress for U_3L_4. The shear in the panel L_3L_4 is—

$$\frac{21895 - 8 \times 287{\cdot}5}{200} = 98, \sec\theta = 1{\cdot}22.$$

The stress in $U_3L_4 = 98 \times 1{\cdot}22 = 119{\cdot}56$ (say 120) units.

Member U_4L_3: The maximum stress in the counter-brace U_4L_3 occurs when wheel 3 is at L_3. The wheel 15 will be over L_0, moving from left to right. The shear in the panel L_3L_4 is for this loading—

$$\frac{13520 - 8 \times 287{\cdot}5}{200} = 56{\cdot}1,$$

and the stress in the counter-brace is 68·5 units—say 69 units.

Member U_2L_3: For this member we must apply the criterion—

$$W - nP\left(\frac{l_1 + g}{g}\right) = \text{a minimum}.$$

$l_1 = 50, g = 225$. The lever arm for the stress in U_2L_3 is 239. Place wheel 3 at L_3, then there will be 29 ft. of uniform load on the bridge. $W = 427{\cdot}5$, $P = 37{\cdot}5$, and $427{\cdot}5 - 8 \times 37{\cdot}5 \times \frac{275}{225} = 61$. If the loads are moved slightly to the left or the right the difference will be greater; hence wheel 3 at L_3 is correct. The reaction at L_0 is—

$$\frac{20455 + 355 \times 29 + 1{\cdot}25 \times 29^2}{200} = \frac{31801{\cdot}25}{200} = 159 \text{ units}.$$

The moments about the intersection of the U_2U_3 and L_2L_3 may be stated as follows:—

$$\text{Stress } U_2L_3 \times 239 - 159 \times 225 + \frac{287{\cdot}5}{25} \times 275 = 0.$$

$$\therefore \text{ The stress in } U_2L_3 = 136 \text{ units}.$$

To find the greatest compression in the bar U_2L_3 the load must be taken as advancing from L_0 towards L_8. Wheel 2 is placed at L_2, and wheel 10 will be 2 ft. in front of L_0. The moment about L_0 is—

$$5790 + 2 \times 190 = 6170.$$

The reaction at $L_8 = \frac{6170}{200} = 30{\cdot}85$, and the panel reaction at L_3 is 4 units. Taking moments about the same point as before we find the stress in U_2L_3—

$$\frac{-30{\cdot}85 \times 425 + 4 \times 300}{239} = -50 \text{ units}.$$

This result is greater than the dead load stress in the member. The tensile stress in the member L_2U_3 may be found by using the lever arm 226 instead of 239—

$$\frac{50 \times 239}{226} = 53 \text{ units}.$$

Member U_1L_2: The criterion $W - nP\left(\frac{l_1 + g}{g}\right) = \text{a minimum}$ is satisfied when the wheel 3 is at L_2, and there is 54 ft. of uniform load on the bridge; $l_1 + g = 140$, $g = 115$, and the lever arm of the stress in U_1L_2 is 123. The reaction at L_8 is—

$$\frac{20455 + 355 \times 54 + 1{\cdot}25 \times 54^2}{200} = 216{\cdot}4 \text{ units}.$$

The panel reaction at L_1 is 11·5 units.

$$\text{The stress in } U_1L_2 = (216\cdot4 \times 115 - 11\cdot5 \times 140)\frac{1}{123}$$
$$= 189\cdot2 \text{ (say 190) units.}$$

The maximum compressive stress in U_1L_2 is found by taking the loads as advancing from left to right, with wheel 2 at L_1 and wheel 6 one foot in front of L_0. The reaction at L_8 is found to be 10·89, and the panel reaction at L_2 is 4 units; then—

$$-123\,U_1L_2 - 10\cdot89 \times 315 + 4 \times 165 = 0.$$
$$\therefore U_1L_2 = -22\cdot6 \text{ (say 23) units.}$$

This result, added to the impact effect, is less than the dead load stress, so that no counter-brace is necessary.

Member U_2L_2: The moment point is 115 ft. from L_0; $l_1+g=165$, $l_1 = 50$. The lever arm of the stress in U_2L_2 is 165. The stress in U_2L_2 is—

$$-\left\{159 \times 115 - \frac{287\cdot5}{25} \times 165\right\}\frac{1}{165} = -100 \text{ nearly.}$$

There is also a tensile stress developed in this member when the load advances from L_0 towards L_8, due to the difference between the vertical components of the stresses in U_1U_2 and U_2U_3. The maximum value of this tensile stress occurs when the live load compressive stress in the inclined member U_2L_3 is just equal to the dead load tensile stress. Messrs. Burr and Falk[1] have shown how to obtain this stress by means of influence lines. In Fig. 392 an equivalent uniform load is substituted for the wheel concentrations, extending from L_0 to the point ϵ. The point ϵ is determined graphically in the manner already explained. A load equal to unity is taken at L_2, and the stress in U_2L_3 is $R_2 \sec\theta$, or $0\cdot25 \times 1\cdot27 = 0\cdot32$. Draw oq to scale equal to 0·32, Fig. 393, and complete the influence line $Sq\epsilon'u$T as shown. The stress in U_2L_3 for a uniform load equal to 3 units per ft. is equal to the area of the triangle $Sq\epsilon' \times 3$. The distance $L_2\epsilon$ or $o\epsilon'$ is about 10 feet; then—

$$\frac{60 \times 0\cdot32}{2} \times 3 = -28\cdot8 \text{ units.}$$

This value is not quite equal to the dead load stress of 35 units, but considering that a uniform load was used instead of the actual wheel loads the position of the load causing zero stress in U_2L_3 may be taken as sufficiently exact. Assuming, therefore, that there is no stress in U_2L_3, the stress in U_2L_2 may be found by means of the influence line

---

[1] *Influence Lines*, by Messrs. Burr and Falk.

S*ph*T. The stress in U_2L_2 is first determined for unity load at L_2, and it may be shown that this stress is—

$$\frac{R_1 l_1 d}{U_2L_2 \times p}$$

where R_1 = reaction at $L_0 = 0{\cdot}75$. $l_1 = 50$ ft., $d = (33 - 28) - (36 - 33) = 2$, $U_2L_2 = 33$ ft., $p = 25$ ft. The stresses in U_2L_2 for unity load at L_2 is therefore—

$$\frac{{\cdot}75 \times 50 \times 2}{33 \times 25} = \frac{1}{11}.$$

Make $op = \frac{1}{11}$ and draw the influence line for U_2L_2, and find the area $Sph\epsilon'$ equal to 3·14. The stress in $U_2L_2 = 3{\cdot}14 \times 3 = 9{\cdot}42$ units. In a similar manner the stress in U_3L_3 may be determined. It will be observed that these stresses do not correspond with those written on the members U_2L_2 and U_3L_3, Plate III. The live load stress in the member U_1L_1 is due to the maximum reaction of the cross-girder attached to it, and is found to be exactly the same as in the Pratt truss, Plate II., or 95 units.

Member L_0U_1. The stress in this member occurs when the reaction at L_0 is a maximum, and this occurs when wheel 5 is at L_0 and there is 89 ft. of uniform load on the bridge.

The reaction at L_0 is—

$$\frac{20455 + 355 \times 89 + 1{\cdot}25 \times \overline{44{\cdot}5}^2}{200} = 272{\cdot}6.$$

The stress $L_0U_1 = 272{\cdot}6 \times 1{\cdot}34 = 365$ units.

The live load stresses written on the outline drawing of the truss, Plate III., have been increased for the impact effect by using the formula—

$$i = l\left(\frac{300}{L + 300}\right)$$

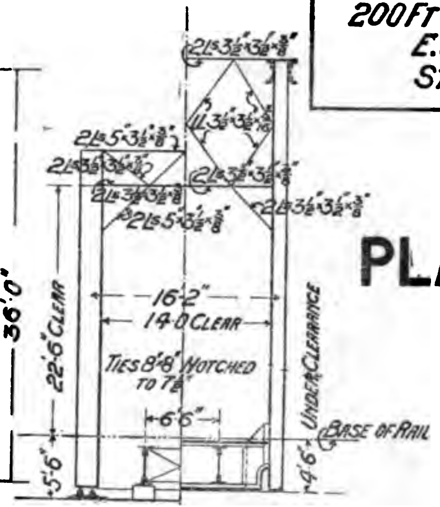
where l denotes the live load stress, and L the length of the span covered by the moving load producing the stress. For the top and bottom chords, $i = 0{\cdot}6l$. For the member U_1L_2 the length of the load is 163 ft. and $i = 0{\cdot}645l$.

For the member U_1L_1. $L = 50$, and $i = 0{\cdot}857l$. See also Table LXX. used in connection with the Pratt truss. The design of the various details in this bridge is illustrated in Plate III.*a*. The stresses due to wind and the design of the wind bracing are more or less similar to the cases fully treated in connection with the Pratt truss railway bridge and may be left to the student to complete.

AMERICAN BRIDGE COMPANY
STANDARD SINGLE TRACK
THROUGH PIN CONNECTED SPAN
200 FT C TO C END PINS
E.50 LOADING
STRESS SHEET

H.J.S.

PLATE ·III·


$2L^s\ 3\frac{1}{2}"\times3\frac{1}{2}"\times\frac{3}{8}"$
$1L\ 3\frac{1}{2}"\times3\frac{1}{2}"\times\frac{5}{16}"$
$2L^s\ 5"\times3\frac{1}{2}"\times\frac{3}{8}"$
$2L^s\ 3\frac{1}{2}"\times3\frac{1}{2}"\times\frac{5}{16}"$
$2L^s\ 3\frac{1}{2}"\times3\frac{1}{2}"\times\frac{3}{8}"$
$2L^s\ 3\frac{1}{2}"\times3\frac{1}{2}"\times\frac{3}{8}"$
$2L^s\ 3\frac{1}{2}"\times3\frac{1}{2}"\times\frac{3}{8}"$
$2L^s\ 5"\times3\frac{1}{2}"\times\frac{3}{8}"$
36'0"
22'6" CLEAR
5'6"
16'2"
14'0" CLEAR
TIES 8'8" NOTCHED TO 7½"
6'6"
UNDER CLEARANCE
4'6"
BASE OF RAIL

HALF END VIEW — HALF INT. SECTION

STRINGERS 25'-0" C TO C FLOOR BEAMS

WEIGHT OF TRACK = 400# PER LIN. FT. = 5000# PER STRINGER
" " 1 STRINGER 4500
9500

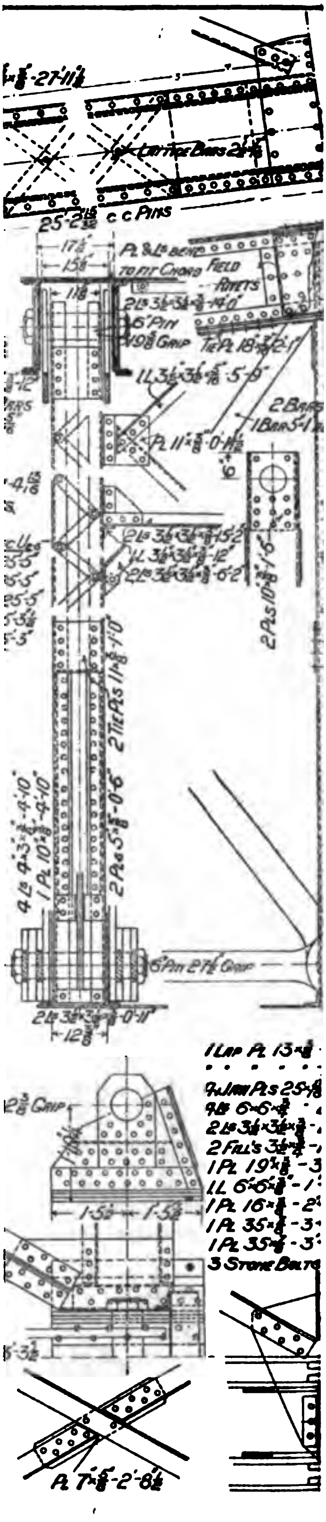
MAX SHEAR 5000# D MAX MOMENT 356000"# D
71000# L 4575000"# L
66000# i 4223000"# i
$142000\times\frac{11.8}{14.4}=116000$ 9154000"#
FLANGE STRESS = 9154000 ÷ 37.8 = 242200#
" AREA = 242200 ÷ 17000 = 14.2□"
$2L^s\ 6\times6\times\frac{9}{16}$ = 12.8□" GR. = 11.8 NT. 12 RIVETS REQ. THRO' FLANGE IN D. OF WEB
$\frac{1}{8}$ WEB 2.6 28 FIELD RIVETS REQ. IN END CONN
14.4
WEB PLATE $41"\times\frac{1}{2}"$ = 20.5□" NO INT. STIFFS. END CONN. $L^s\ 5"\times3\frac{1}{2}"\times\frac{7}{16}"$

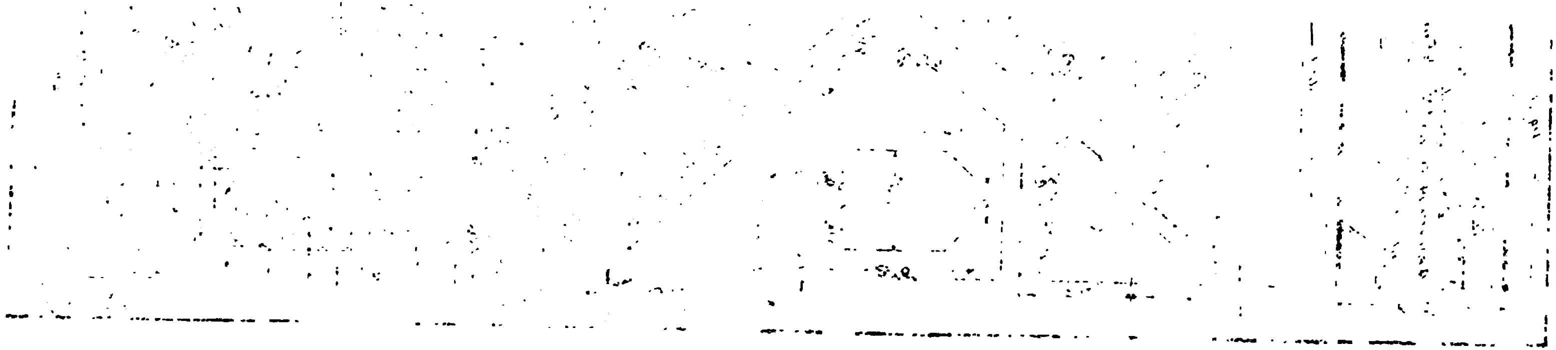
INTERMEDIATE FLOOR BEAMS 16'-2" C TO C TRUSSES

MAX SHEAR 11000# D MAX MOMENT 638000"# D
95000# L 5510000"# L
81000# i 4690000"# i
$187000\times\frac{9.5}{12.8}=139000$ 10838000"#
FLANGE STRESS = 10838000"# ÷ 50.2 = 215000#
AREA = 215000 ÷ 17000 = 12.6□"
WEB PL $52"\times\frac{1}{2}"$ = $26\times\frac{1}{8}$ = 3.3□" NT. 17 RIVETS REQ THRO' FL. BET. PST & STR.
$2L^s\ 6\times4\times\frac{9}{16}$ = 10.6□" GR = 9.5 = 36 FIELD RIVETS REQ IN END CONN
12.8 END CONN. $L^s\ 5\times3\frac{1}{2}\times\frac{7}{16}$

END FLOOR BEAMS 12'-4" B. TO B. OF L^s

MAX SHEAR 6000# D MAX MOMENT 210000"# D
77000# L 2690000"# L
71000# i 2480000"# i
$154000\times\frac{6.4}{9.2}$ 112000 5380000"#
FLANGE STRESS 5380000 ÷ 56.8 = 94600#
AREA 94600 ÷ 17000 = 5.6□"
WEB PL $58\frac{1}{2}\times\frac{3}{8}$ = $22□"\times\frac{1}{8}$ = 2.8□" NET
$2L^s\ 6\times4\times\frac{3}{8}$ = 7.2 GR. = 6.4□" = END FLOOR BEAM TO BE USED ONLY WHEN SPECIFIED
9.2


×⅝-27'-11½"
Lattice Bars 2½×⅝
25'-2 15/32" c c Pins
17½"
15⅝"
11⅛
Pl. & L's bent to fit Chord
Field Rivets
2 L's 3½×3½×⅜-14'-0"
6" Pin
9⅝ Grip
Tie Pl. 18×⅜-2'-1"
1 L 3½×3½×5/16-5'-9"
2 Bars
1 Bar 5×1
Pl. 11×⅜-0'-11½"
6"
-4 15/16
2 L's 3½×3½×⅜-15'-2"
1 L 3½×3½×⅜-12"
2 L's 3½×3½×⅜-6'-2"
2 Pl's 10×⅜-1'-6"
2 Tie Pl's 11×⅜-1'-0"
4 L's 4×3×½-4'-10"
1 Pl. 10×⅜-4'-10"
2 Pl's 5×⅜-0'-6"
6" Pin 2 7/8" Grip
2 L's 3½×3½×⅜-0'-11"
12⅜
1 Lap Pl. 13×⅜
2⅜ Grip
1'-10¼"
1'-5½"
1'-5½"
4 Jaw Pl's 25×⅜
4 L's 6×6×⅜
2 L's 3½×3½×⅜
2 Fill's 3½×⅜
1 Pl. 19¼×⅜ -3
1 L 6×6×⅜ -1
1 Pl. 16×⅜ -2
1 Pl. 35×⅜ -3
1 Pl. 35×⅜ -3
3 Stone Bolts
5'-3½
Pl. 7×⅝-2'-8½"



S.R.

CHAPTER XX.

SWING BRIDGES.

Among the most important class of movable bridges are those known as swing bridges. These may be classified according to the method adopted for supporting their weight during the process of swinging; thus—

(*A*) Bridges entirely supported on rim-bearing turn-tables consisting of a circular roller path, with a ring of conical rollers, upon which rests a circular girder carrying the main girders of the bridge.

(*B*) Bridges supported on a centre-bearing turn-table, consisting of a central pivot, or a nest of one or more series of conical rollers, with or without a ring of live rollers, or two or more wheels for steadying the bridge during the process of opening. The weight may be proportioned in this class of bridge so as to partly rest on the central pivot and partly on the live rollers.

(*C*) Bridges which are partially or completely lifted by means of a central hydraulic press, and turn on a water centre or on a spherical pivot.

(*D*) Floating swing bridges, in which the load upon the ring of live rollers or central pivot is buoyed up by means of a hollow caisson which carries the bridge, so that only a portion of the load rests upon the pivot or rollers.

Class A.—This class of bridge is constructed with two cantilevers of equal or unequal lengths; in the former case one arm balances the other, in the latter the shorter arm must be loaded so as to balance the longer.

An example of this kind of bridge occurs at Goole over the river Ouse,[1] in which the two arms are each 100 ft. in length.

Other examples exist at the entrance from Alfred Dock to East Float, Birkenhead, the entrance to Alfred Dock,[1] the river Hull south bridge,[1] and in numerous highway bridges over navigable rivers. The bridges mentioned are moved by hydraulic power, but the special design of this class of bridge has very little to do with the motive power used for swinging it, whether hydraulic, steam, or hand power.

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[1] Vol. lvii., *Proc. Inst. C.E.*, paper by J. Price, M.Inst.C.E.

It frequently happens that hand power is the only means available for rotating the bridge.

It is important in this class of bridge that the weight should be equally distributed over the live rollers, so that they may work with the minimum friction. The main girders generally rest upon the circular girder at four points, which arrangement does not distribute the weight with sufficient equality over the rollers. It is much better to divide the weight equally over six or more equidistant points. This matter will be further illustrated.

Class B.—This class includes a large number of important swing bridges. The central pivot may be arranged to carry the whole of the weight, but the more usual proportion is from two-thirds to three-quarters.

The bridge over the Raritan[1] erected by the Keystone Bridge Company, Philadelphia, from the designs of Mr. J. H. Linville, is 472 ft. long, with two passages each 216 ft.

The total dead load is 4500 lbs., and the live load 2500 lbs. per foot run. The load upon the central pivot may be regulated to any desired extent by means of the suspension bolts. Before being swung, the bridge is lifted by four hydraulic rams to a height of 4 in. clear of the abutments, and while being closed it is lowered directly on to the abutments, so that wedges or other modes of setting up are unnecessary.

The bearings are adjusted to such a height that each carries half the weight of the bridge; the lower chords are parted in the centre, and a slotted link connection is used in the top chord, so that the bridge acts as two cantilevers while being swung, and as two independent girders when resting on the abutments. In this way the disadvantages of a long continuous girder are overcome, and the end does not rise when a heavy train approaches from the other end of the bridge, producing a hammer-like action of the main girder on the abutments. To overcome this difficulty, latching apparatus is sometimes provided for fixing the ends when the bridge is closed.

The Kansas City bridge consists of two openings each 160 ft., and is set up on the abutments by means of sliding wedges. Both these bridges are moved by means of steam power. The Hawarden bridge over the river Dee, England, includes a swing span 284¾ ft. long, forming openings 87 and 140 ft. long respectively.[2] The main girders are of steel 32 ft. deep over the centre of the swing pier, 9½ ft. deep at the ends. The girders are lattice, divided into panels 17 ft. long, with single triangulated vertical and diagonal members, except over the pivot, where the members cross each other. The top chords are curved and the bottom chords horizontal, of trough-shaped section. The cross-girders are spaced 17 ft. centres, and are 29 in. deep over the angle irons. The weight per sq. ft. of the deck is 38 lbs., including brackets and gangway plates, but exclusive of the permanent way. The main girders were built without camber, and provision was made in the lifting apparatus at the tail end for taking out the deflection which occurs when the longer end is unsupported, so that the stresses in the girder due to its own weight are removed as much as possible.

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[1] Vol. lvii., *Proc. Inst. C.E.*, paper by J. Price, M.Inst.C.E.

[2] Vol. cviii., *Proc. Inst. C.E.*, paper by F. Fox, M.Inst.C.E.

The weight of the tail end is made 15 tons heavier than the longer end by loading it, and the load upon the central pivot is regulated by means of suspension bolts acting on the pivot cross-bearing girder which supports the bridge over the pivot pier, so that the whole or any portion of the total load of the swing span may be thrown upon the central pivot. The bridge has a slight bascule movement for the purpose of lifting the fore ends out of their bearing sockets, which is effected by lowering the tail ends by means of two rams, each 25 in. in diameter and 8¾ in. stroke, capable of supporting and lifting 300 tons. Four rollers of cast steel 3 ft. in diameter are provided, two under each end of the main bearing girder, and directly under the main girders on the pivot pier. The rollers revolve in wrought-iron carriages upon fixed radial axles. Two cast-steel rollers, also 3 ft. in diameter, are arranged at the tail end, one under each main girder, which carry the preponderating load only, *viz.*, 15 tons.

A solid bearing pivot was adopted in preference to a hydraulic ram or water-bearing, as the necessary vertical movement of the bridge being provided for, a ram was not necessary. The cup-and-ball pivot is 30 in. in diameter, of forged wrought-iron, case hardened, which provides for both rotating and bascule movement. The lubricant used is heavy cylinder oil, which is injected into the bearing about once for every three swings. The weight resting on the pivot is 732 tons while the bridge is being turned, and the average pressure per sq. in. about one ton; the maximum pressure at the centre is, however, much greater, as the radius of the cup is ¼ of an in. larger than that of the ball.

The bridge is moved by two single-acting hydraulic rams securely attached to the solid brickwork of the main pier. The slewing drum is 32 ft. in diameter, and forms a part of the main bearing girder. The tail-end rollers are fitted with a winch gear and brake, in order that the bridge may be opened and closed by hand. The bascule movement of the bridge is utilised in mechanically locking it in position when closed, and for closing the space which would otherwise occur between the ends of the rails on the moving and fixed portions of the bridge. The important moving parts of the bridge are interlocked with the signals on the railway. The opening of the bridge requires the following operations: (1) Lifting the tail end of the bridge $\frac{1}{8}$ in. by the tail-end rams; (2) withdrawing the sliding bearing blocks from under the tail ends; (3) lowering the tail-end rollers; and (4) setting in motion the rotating rams.

Class C.—This class of bridge, which is made to turn entirely on a long central pivot of conical form, is used largely in Holland. It is necessary for the roadway to be sufficiently high above the water-level to allow of the necessary length of the pivot, and also for the main girders to be arranged below the roadway, keeping the centre of gravity below the point of support. The foundations must be good, as any settlement might cause the bridge to jam so that it could not be turned.

Class D.—This class of bridge is used largely in connection with the entrances to docks. The bridge is arranged so as to be out of balance, and is supported in the centre upon an hydraulic press and at the tail end upon wheels. When the centre is raised, the bridge tilts over until the wheels on the tail end come in contact with the tram plate. The tail end may tilt upwards or downwards; in the former case the tram plate is inverted. The central pivots are made spherical, but the bridge actually turns on a water-centre. Examples of this class of bridge exist at the Albert Dock, Hull; the West India Docks, London; the Leith bridge, England; and the Marseilles bridge, France.

Another example of this class of bridge is the swing bridge of Le Pollet (Dieppe).[1] This bridge was necessitated by the new dock works, Dieppe, and crosses a waterway 131 ft. wide. The longer arm of the bridge, when swinging, is 154 ft. and the shorter arm 77 ft. The main girders are 30 ft. deep over the pivot, with horizontal bottom booms and curved top chords of trough-shaped section. The panel length is 16 ft. with vertical posts and diagonals, in both directions. The total weight of the bridge swinging is 800 tons, and the ballast weighs 234½ tons, or 20 tons more than is necessary to counterbalance the bridge. The deflection of the long end in swinging is $4\frac{1}{8}$ in. The bridge, when in use, rests on three sets of bearings, *viz.*, at the tail end and on the passage walls, being clear of the pivot. When turning it is carried on a hydraulic plunger, and the excess weight at the tail end rests by two pairs of rollers on the rail track. The opening of the bridge requires the following operations: (1) The cylinder under the pivot is opened to the pressure water, and the plunger-head brought in contact with the bridge. The bridge does not yet lift, as the area of the plunger is only proportioned to 624 tons. (2) Pressure is opened in two hydraulic cylinders placed one under each main girder near the tail end of the bridge. The bridge is now lifted from its bearings and is water-borne on three cylinders. (3) The supports under the tail end are withdrawn by means of a small hydraulic cylinder operating through a crank and connecting rod. (4) Pressure is withdrawn from the two lifting cylinders at the tail end. The bridge now falls again, and, owing to the overbalance, tilts backwards until the rollers at the tail end rest on the rail, the central plunger sinking to the bottom of its cylinder, where it rests upon a steel ring let into the bottom of the casting. (5) The bridge is now swung by the usual hydraulic arrangements. The weight is thus divided in turning between the water pressure, which carries 624 tons, the steel ring in the cylinder carrying 156 tons, and the rail track at the tail carrying 20 tons. In closing the bridge these operations are reversed. The opening or closing takes usually from two to three minutes, although it can be done in one minute and a half.

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[1] *Annales des Ponts et Chaussées*, December, 1891, p. 584; abstract, vol. cix., p. 416, *Proc. Inst. C.E.*

The large swing bridge over the Tyne, at Newcastle, cannot be included strictly in any of the foregoing classes. This bridge is supported and turned on a water-centre at a constant level, acting as a relieving, not a lifting, pivot. It is in every respect a most satisfactory structure.

Modern swing bridges are constructed with girders or trusses, supported on centre-bearing, or rim-bearing, turntables. Sometimes the bridge may be partly supported on a central pivot and partly on a rim-bearing turntable. The girders and trusses used in connection with centre-bearing turntables are continuous over the central support, forming a two-span continuous bridge. In the rim-bearing turntable the bridge may be continuous over two points of support. In all such cases the reactions, moments and stresses may be found in the manner explained in Chapter XI.

The centre-bearing turntable is frequently used with plate web girders or lattice girders, or trusses; the ends may be latched down, or lifted when the bridge is closed. The latching apparatus is capable of providing both positive and negative reactions; the ends are prevented from rising from their points of support, and hammering of the ends is prevented. It is, however, more usual to lift the ends of the bridge to prevent hammering, and the upward forces applied should be at least equal to the negative reactions determined for latched ends. The stresses in a swing bridge with lifted ends should be found for the following cases:—

1. The dead load stresses when the bridge is acting as a cantilever during swinging.
2. The stresses caused by the required lift at each end.

3. The live load stresses when one arm is loaded considered as a continuous girder, or when both arms are loaded.
4. The live load stresses when one arm is loaded considered as a simple span.

The maximum stresses are usually found by combining the stresses found for 1, 2, the latter portion of 3, and 4.

The rim-bearing turntable is usually adopted in the case of trusses, and these may be continuous over two points of support, or partially continuous, and in some cases non-continuous, in which the two end spans are simple trusses.

Fig. 394 shows a swing bridge in which the truss is continuous over the centre span *ef*. The diagonals *ed* and *cf* must be made of sufficient sectional area to ensure complete continuity. When the bridge is open the stresses in each arm are found as in a cantilever.

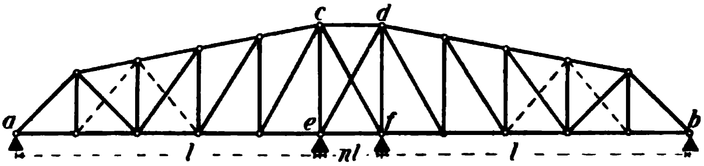


FIG. 394.

When the bridge is closed and the ends locked the dead load stresses are not altered, and the live load stresses are determined as in a continuous girder of three spans. If the ends are lifted sufficiently to ensure that the dead load stresses are the same as in an ordinary continuous girder, then the live load stresses must be combined with these dead load stresses to find the maximum and minimum stresses. It has been pointed out by Prof. Johnson that the reactions at *e* and *f*, due to the live load on *ae* or *fb*, when treated as a three-span continuous girder, are greater than the actual reactions at these points. The error is due to the ordinary continuous girder treatment, which assumes a constant moment of inertia and neglects the effect of the shearing stresses in producing deflection. For beams and long-span trusses the error is not important, but in short-span trusses, as used in swing bridges, the error may be considerable. The true reaction at *f* for the span *fb* loaded with the live load may be found by calculating the deflection of the point *c* in the direction *cf* when both diagonals are absent. The elongation of the diagonal *cf* cannot exceed this movement, and may be less if the sectional area of the member is large. The stress due to initial tension must be added to the stress due to the elongation of *cf*, due to the deflection of the point *c*. The reaction at *f* is the stress in *df* minus the vertical component of the stress in *cf*.

Fig. 395 shows a partially continuous swing bridge suitable for a large span, in which the dotted members forming the bracing of the central tower is made merely strong enough to stiffen the towers but are not sufficient to carry the shearing stresses due to continuity. The theorem of three moments cannot be applied in this case, but the reactions may be approximately determined such that the internal work of the stresses will be a minimum. The sum of the reactions must equal the load, and the sum of the moments of the reactions equals the moment of the load, or—

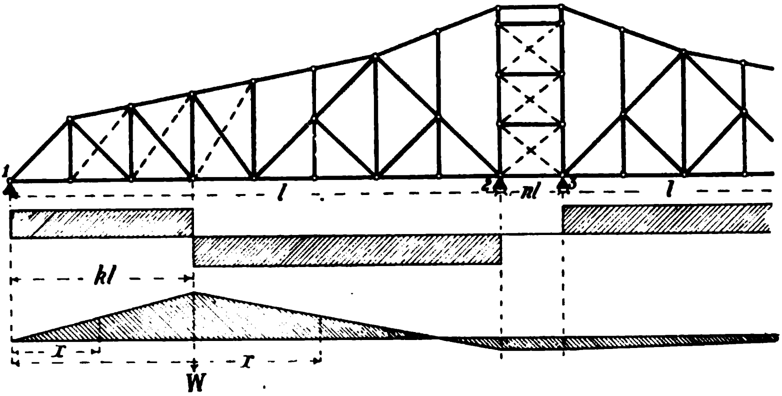


FIG. 395.

$$R_1 + R_2 + R_3 + R_4 = W\,;\quad R_4(2l + nl) + R_3(l + nl) + R_2 l = Wkl.$$

Also, since there is no shear in nl—

$$R_1 + R_2 = W.$$

It follows, therefore, that—

$$R_2 = W - R_1\,;\ R_4 = -R_3 = R_1 - W(1 - k).$$

Now since the work of the internal stresses is proportional to the sum of the squares of all the bending moments, and since these are for a load W in the first span at a distance kl from 1—

R_1x between W and 1, $R_1x - W(x - kl)$ between W and 2. R_4l between 2 and 3, and R_4x between 3 and 4. The work of the internal stresses is—

$$\int_0^{kl} R_1^2x^2dx + \int_{kl}^{l} (R_1x - Wx + Wkl)^2dx + \int_0^{nl} R_4^2l^2dx + \int_0^{l} R_4^2x^2dx,$$

and the values of R_1 and R_4 must be such as to make this a minimum. Integrating and substituting for R_4 in terms of R_1, and making this a minimum in regard to R_1 we obtain—

$$R_1 = W(1 - k) - \frac{W}{4 + 6n}(k - k^3)$$

$$\therefore R_2 = Wk + \frac{W}{4 + 6n}(k - k^3)$$

$$R_3 = - R_4 = \frac{W}{4 + 6n}(k - k^3).$$

When the stress sheet and the sectional areas of the bridge have been determined we can find the reactions in the following manner :—

Determine the deflection of the end of the truss when it is supported at the points 2, 3 and 4, the point 1 being unsupported, by the formula in Chapter X.—

$$v = \frac{1}{WE}\Sigma\frac{S\sigma l}{A} = \Sigma\frac{S\sigma l}{AE}.$$

The summation must extend from 1 to 4.

Where W = 1 ton applied at the point 1, S = the stress in any member of the truss for the loading upon it, and σ = the stress due to W = 1 ton applied at the point 1, A = the sectional area of any member of the truss, l = the length of the member in inches.

If the truss is lifted at the end by the application of a force equal to unity, W = 1, the upward deflection is—

$$V' = - 2\overset{1}{\Sigma}\frac{\sigma^2 l}{EA} \text{ since } S = \sigma.$$

The upward deflection produced by R_1 is—

$$V'' = - 2\overset{1}{\Sigma}\frac{R_1\sigma^2 l}{EA} \text{ since } S = R_1\sigma.$$

Where the summation denoted by $\overset{1}{\Sigma}$ extends from 1 to the centre of 2-3, then—

$$R_1 = \frac{V}{V''} = \frac{\Sigma S\sigma l}{2\overset{1}{\Sigma}\sigma^2 l}.$$

For a load W at a distance of kl from the point 1, the downward reaction at the point 4 is—

$$Rl = W(1 - k)l$$
$$\therefore R = W(1 - k).$$

The portion $\Sigma S\sigma l$ includes the summation from 1 to the middle of the span 2-3, and from the middle of 2-3 to 4; the latter portion may be expressed as follows—

$$\overset{3}{\Sigma}W(1 - k)\frac{\sigma^2 l}{A} = W(1 - k)\overset{3}{\Sigma}\frac{\sigma^2 l}{A}.$$

$$\text{Then } R_1 = \frac{\dfrac{\overset{1}{\Sigma}S\sigma l}{A} + W(1 - k)\overset{3}{\Sigma}\dfrac{\sigma^2 l}{A}}{2\overset{1}{\Sigma}\dfrac{\sigma^2 l}{A}}$$

Where $\overset{3}{\Sigma}$ indicates the summation extending from the middle of 2 - 3 to 4.

When the value of R_1 has been determined the remaining reactions are found in the manner already explained.

Fig. 396 denotes a swing bridge consisting of two simple trusses over the waterway. The centre tower supports a platform carrying a steam engine, arranged to put tension in the members U_5 U_4 and U_6 U_7 and lift the spans. The bridge is then revolved upon the turntable.

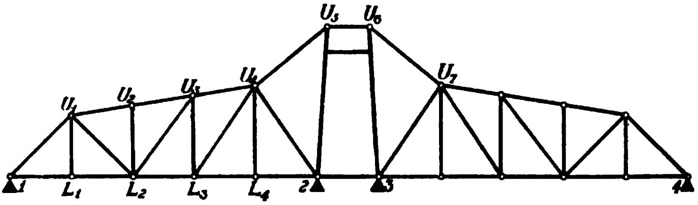


FIG. 396.

In this case there is no ambiguity in regard to reactions; when the bridge is open the upper members U_1 to U_5 are in tension, and the lower members are in compression. When the bridge is closed these stresses are reversed and the two spans behave as simple trusses. The top and bottom members must therefore be designed to resist both tension and compression. Counterbraces should be inserted to come into action when the bridge is traversed by live loads, the members U_5 U_4 and U_6 U_7 being unstressed when the bridge is closed.

In Fig. 397 there is also no ambiguity in reactions or stresses, and no counterbraces are necessary. The inclination of the members U_4 U_3 and U_4 U_5 should be such that, when the bridge is fully loaded, the members will carry the greater part of the load to the points 2 and 3 of the turntable. The link U_4a insures the equal distribution of the loads on the turntable. If the balance is disturbed by impact due to a passing vessel tending to lift the arm on one side, the points 2 and 3 will yield and allow considerable deflection in the arms without moving the truss off the turntable.

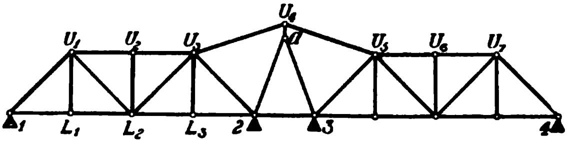


FIG. 397.

Turntable. The best form of turntable is designed in such a manner that a portion of the total load is carried by the central pivot, and the remainder is carried on a ring of live rollers. The load on the pivot fixes the centre of rotation, and relieves the load on the rim so that there is less resistance to turning. Fig. 398 shows a turntable in which the total load is applied to the circular girder *cc* over the live rollers at sixteen points by means of eight distributing girders *aa*, etc. This arrangement ensures easy turning as every roller is in action.

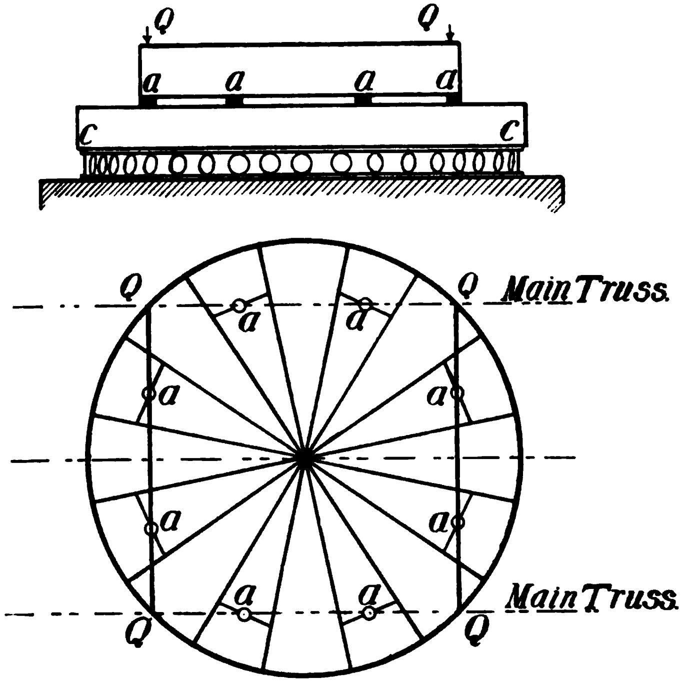
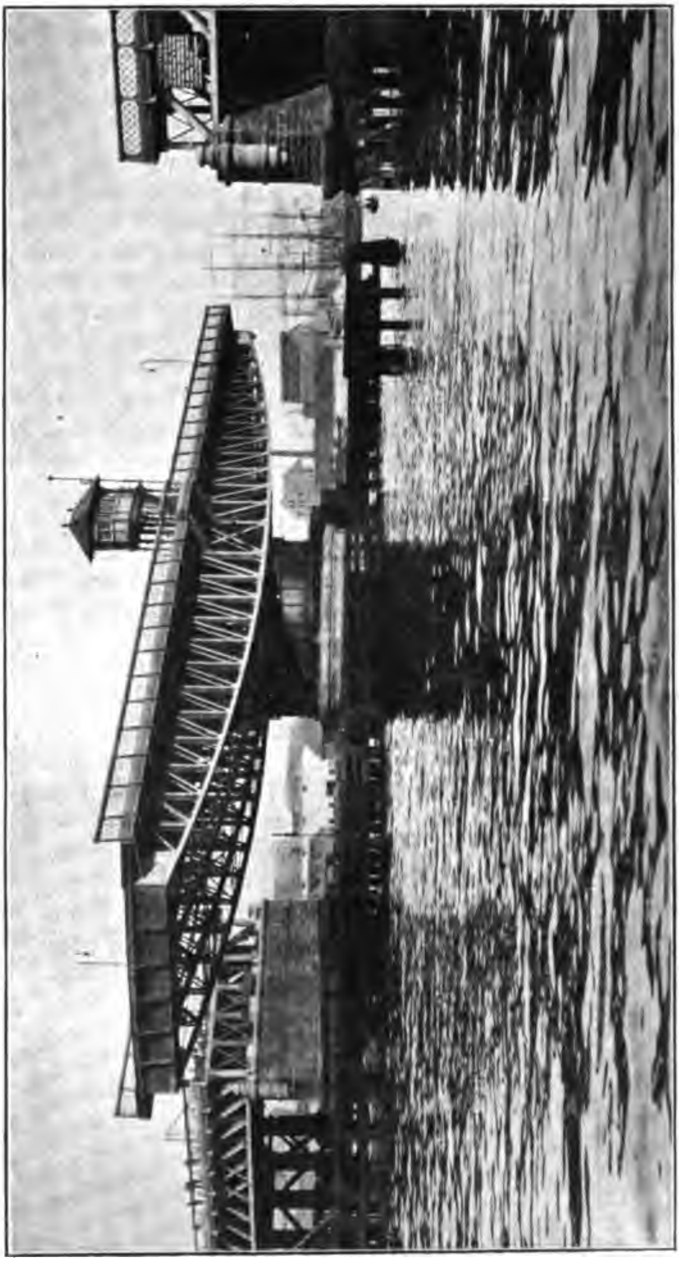


FIG. 398.

Two swing bridges have recently been constructed at Sydney, N.S. Wales, in which the machinery for slewing the swing span, lifting the ends and working the roadway gates is operated by electric motors supplied with electricity from the tramway power-house. The Pyrmont bridge over Darling Harbour has a swing span 223 ft. long (Fig. 399), and consists of four main girders 15 ft. deep at the centre and 5 ft. deep at the ends, spaced 13 ft. 4 in. apart and rigidly braced together over the pivot pier and at the ends. The girders are connected at their upper panel points by cross-girders carrying a rolled joist and buckled plate deck securely attached to the top flange plates of the main girders. The carriageway and the footpaths are covered with wood blocking and asphalt respectively on a bed of coke concrete. The swing is of the rim-bearing type, the whole weight, 850 tons when swinging, being distributed by means of eight small girders



Pyrmont Bridge—Protecting Platform and Dolphins.
Fig. 399.

to sixteen equidistant points on the drum. The swing-span is designed to carry a distributed live load of 100 pounds per sq. ft. of deck, and a concentrated load of 20 tons on a 10 ft. × 5 ft. wheel base. The deflection at the ends of the main girders is 4¼ in. This bridge can be opened or closed in 30 secs., but the usual time occupied is somewhat greater.

The maxima stresses occur over the pivot pier when the bridge is swinging, and the ends are lifted only 1¼ in., so that the dead load taken by the cams at each end of the bridge is about 40 tons. The span was built out from the pivot pier without staging, a stationary crane on the platform alongside raising the material to a crane travelling on the top of the span, by which the different members were placed in position. This bridge is fully described by its designer, Mr. P. Allan, M.Inst.C.E., in a paper read before the Institution of Civil Engineers (*Proc. Inst. C.E.*, vol. clxx.). The calculations of the power required to swing the bridge are also fully described in the correspondence in connection with the above paper by Mr. H. H. Dare, M.E., Assoc.M.Inst.C.E.

Other Forms of Movable Bridges.—It is not proposed in this book to discuss the various types of movable bridges, many of which are very ingenious. A great variety exist in America and Australia. The largest lift-bridge was designed by Dr. Waddell, over the Chicago river, of 150 ft. span and a 50-ft. roadway. The lift is 141 ft. (*Trans. Am. Soc. C.E.*, January, 1895). Several smaller bridges of this class have been built in Australia. The Scherzer rolling lift-bridge has been developed in America, and two examples occur across the Chicago river, consisting of two hinged arms, which are raised by means of electric motors and locked together when the bridge is closed. The Tower Bridge, London, is a fine example of a bascule bridge. Various roller bridges have been built in which the girders are moved in the direction of their length, horizontally, on rollers, suitable for small spans. Other forms consist of two parts which are rolled backwards, and pushed forwards, the two ends being locked together in the centre, and are suitable for larger spans.

CHAPTER XXI.

ARCHED AND SUSPENSION BRIDGES.

THE arch is more economical than the most economical truss or girder, considered merely with regard to the ratio of its carrying capacity to its own weight. This is seen when the weight of a bowstring girder is compared with an arched rib. In the bowstring girder, the curved member must be designed to resist the direct stresses developed in it, and the horizontal member the constant tensile stress. We have seen that for uniform loading, or for equal loads concentrated at each panel-point, there is no stress in the web bracing, but for partial loading produced by a passing train the web bracing will be stressed. The piers supporting the truss must be designed to carry the weight of the structure and the maximum load carried by it. If the ends of the curved member are made to abut against the piers or abutments the horizontal tie may be dispensed with, but the abutments or piers (if there is more than one span) must be made larger, in order that they may be capable of resisting the thrust produced by the end reactions of the curved rib.

If the extra cost of the abutments is less than the cost of the tie, then the arch is a cheaper structure than the truss.

In long spans the saving effected by dispensing with the tie and the bracing bars of the web is considerable, which is due not merely to their extra cost above the extra cost of the abutments and the lighter bracing of the arch, but to the fact that in the arched rib the load sustained is less by the extra weight of material necessary in the bowstring truss.

The suspension bridge is the inverted form of the arched bridge, in which the tension in the suspended rib or cables corresponds with the compression in the arched rib. In either case, if the rib is loaded in any way whatever, the correct form, in order that it may sustain the load without bending, is the equilibrium polygon for the load in question.

Let W_1, W_2, W_3 . . . W_6 (Fig. 400) denote a series of loads acting on a straight beam *amh*, and let *abc* . . . *h* denote the equilibrium polygon for these loads obtained by drawing lines parallel to the strings in the force polygon, Fig. 401. Draw the line H perpendicular to the load line, Fig. 401, representing to the scale of the force polygon the horizontal component of the stress in *ah*.

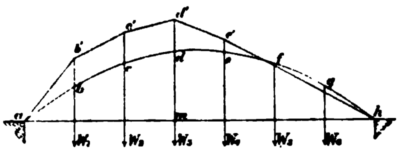


FIG. 400.

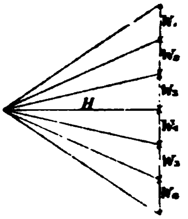


FIG. 401.

Then, by Culman's principle, the bending moment at any point in the beam, such as *m*, is equal to—

$$dm \times \text{H}.$$

If we denote the bending moment at any point by $\Sigma \text{W}x$, and the corresponding ordinate of the equilibrium polygon by V, then—

$$\Sigma \text{W}x = \text{HV, and V} = \frac{\Sigma \text{W}x}{\text{H}}.$$

From this equation we observe that the height of the equilibrium polygon is inversely proportional to H.

It may be shown that the form of the equilibrium polygon for loads distributed equally along the horizontal is a parabola, for loads equally distributed along the length of the rib it is a catenary, and for uniform pressure acting normally at any point in a curve it is a circle.

If the form of the arched rib corresponds with the polygon *abc* . . . *h*, there will be no bending stress upon it; but if the system of loading produces an equilibrium polygon *ab'c'd'e'h*, the arched rib will be subjected to bending stress.

Let V′ denote the ordinate of the polygon, and V that of the rib, and M the bending moment at the point *m*; then—

$$\text{M} = \Sigma \text{W}x - \text{HV}'$$

$$\text{but since } \Sigma \text{W}x = \text{HV}$$

$$\text{M} = -\text{H}(\text{V}' - \text{V}).$$

The bending moment at any point of an arched rib for any system of loading is proportional to the difference between the ordinates of the rib and of the equilibrium polygon for that system of loading.

This is perfectly general, and applies to arches with fixed and free ends.

The loads upon an arched rib produce at any section—

(1) A normal thrust which is uniformly distributed over the area of the rib at the section, its horizontal component being H.

(2) A shearing stress at the section.

(3) A bending moment H(V′ − V) producing tension and compression on opposite sides of the neutral axis of the section.

The direct compression, combined with bending, produces a distribution of stress over the section similar to that which has been shown to exist in short columns. The methods adopted in the foregoing investigations of the two-hinged arched rib, also the arched rib without hinges, is partly taken from Professor Burr and Professor Eddy.[1]

Let S = the shearing stress at any section.

W = any load upon the rib.

M = the bending moment at any section.

V = the vertical deflection of the rib at any section.

V_h = the horizontal deflection.

a = the strain, extension, or compression in any fibre situated at a perpendicular distance unity above or below the neutral axis and parallel to it.

n = the length of the parts into which the arched rib is divided; the smaller the value of n the more accurate the results.

Then the following approximate equations may be used:—

$$S = \Sigma W \qquad (1)$$

$$M = \Sigma Wx \qquad (2)$$

$$a = \frac{M}{EI} \qquad (3)$$

$$\Sigma na = \Sigma\frac{nM}{EI} \qquad (4)$$

$$V = \Sigma nax = \Sigma\frac{nM}{EI}x \qquad (5)$$

$$V_h = \Sigma nay = \Sigma\frac{nM}{EI}y \qquad (6)$$

From (2) and (5) we have—

$$M : V :: \Sigma Wx : \Sigma\frac{nM}{EI}x$$

$$:: \Sigma W : \Sigma\frac{nM}{EI}$$

Hence in any equilibrium polygon in which $\frac{nM}{EI}$ is used instead of W, the ordinates will represent the deflections.

In an arched rib, with fixed or free ends, the depth should be made sufficient to enable it to retain its form under the partial distribution of the maximum live load. This is most conveniently done by dividing the material of the rib into two portions, placed one below the other, forming two arches united together with suitable bracing. The finest example of the rigid arch occurs in the bridge over the Mississippi at St. Louis (see Fig. 402), which consists of three spans, 502, 520, and 502 ft. Each span consists of four ribs, and each rib is constructed with two steel tubes spaced vertically 12 ft. apart centre to centre and united with bracing. The dead load per lineal ft. is 1 ton on each rib, and the live load produced with a crowd of people on a roadway and footpaths 52 ft. wide, and two lines of rails underneath loaded with engines, is 0·8 ton per foot run on each rib.

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[1] *Stresses in Bridge and Roof Trusses*, by W. H. Burr; *Researches in Graphical Statics*, by H. T. Eddy.

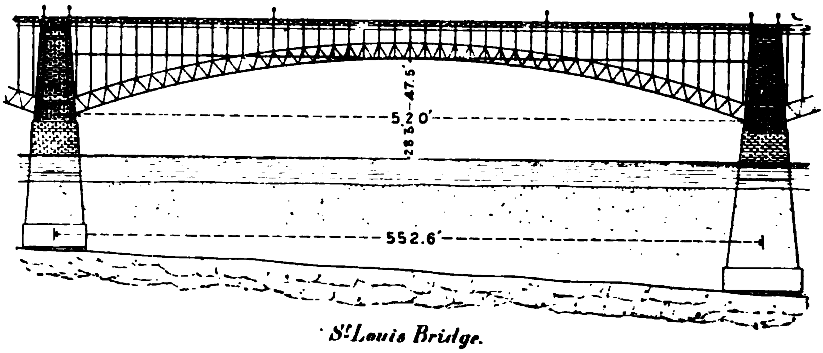


FIG. 402.

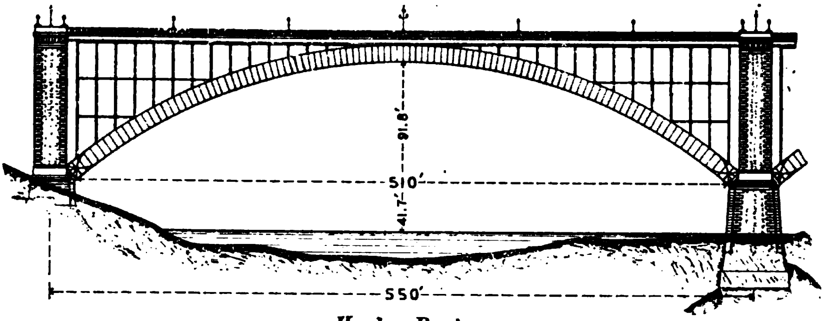
The total weight of iron and steel in one rib of the central span is 218 tons. The total weight of iron and steel in the four ribs, including vertical struts to roadway, and railways, wind bracing, etc., is about 1600 tons. The total cost of the entire bridge was £1,361,873.

Four bowstring trusses of the same central span as the St. Louis bridge would weigh at least 500 tons more. The temperature stresses were calculated for an extreme variation of ± 80° Fahr. The maximum working stress allowed was 13·4 tons in compression, and 8·9 tons in tension.

As an example of an arched bridge hinged at the springing, we may select the Harlem River bridge (Fig. 403), which consists of two steel arches, each 510 ft. span, with masonry piers and abutments. Each span is constructed with six steel ribs spaced 14 ft. centres under the roadway, which is 80 ft. wide, with two footpaths. The effective depth of the arched ribs is 12 ft. The bridge was designed for the following loads :—

Floor of Bridge.—A dead load 225 lbs. per sq. ft.
A live load 100 „ „
Or a 20-ton steam roller.

Arched Ribs.—A dead load of each rib 1·1 ton per foot run.
A dead load of floors and paving 1·35 ton per foot run.
A live load of 0·6 ton per foot run.
A wind pressure of 5·4 tons per foot distributed in proportion to the exposed surfaces.



Harlem Bridge.
FIG. 403.

The modulus of elasticity of the iron was assumed to be 26,000,000 lbs. per sq. in. Working stress in ribs, tension or compression, on gross area, 6·7 tons per sq. in.; on net sections, 8 tons per sq. in. Extreme variation of temperature, ± 75° Fahr.

Arched Rib Hinged at the Springing.—The equilibrium polygon for any loads acting on an arched rib hinged at the springing and continuous at the centre may be determined in the following manner:—

Let Ad_1, d_2 . . . B, Fig. 404, denote the neutral axis of the rib which is 147 ft. span, with a central rise of 20·5 ft.

Let the centre line of the rib be divided into a number of equal parts, the length of which may be denoted by n, representing the points of application of the loads upon the rib; the smaller the distance between these divisions the more nearly will they represent a uniform load, and if the points of application are taken sufficiently near to each other, the distance n may be considered to have the same value whether measured horizontally or along the curve of the rib. We may consider the rib as one of a pair, spaced 16 ft. apart centre to centre, in a bridge carrying a roadway 18 ft. wide between the kerbs. The deck may be supposed to be secured to two longitudinal girders supported at the panel points of the rib by means of vertical columns. Let the dead load concentrated at the panel points of the ribs, 7·35 ft. apart, be 6 tons, and the live load equivalent to 80 lbs. per sq. ft. of roadway be 2·6 tons. Only one case of loading will be considered, *viz.*, that in which each panel point on the left half of the rib is loaded with the live and dead load, or 8·6 tons, and each panel point on the right half with 6 tons; the centre panel point is loaded with $\frac{6 + 8\cdot6}{2} = 7\cdot3$ tons. As a matter of fact, the dead load is not uniformly distributed horizontally, being greater towards the abutments. Draw the equilibrium polygon for this distribution of loads, $Ga_1a_2a_3$ . . . H, Fig. 405, with an assumed polar distance EF, Fig. 406, setting off the loads on the vertical line XLY, Fig. 406, and draw the closing

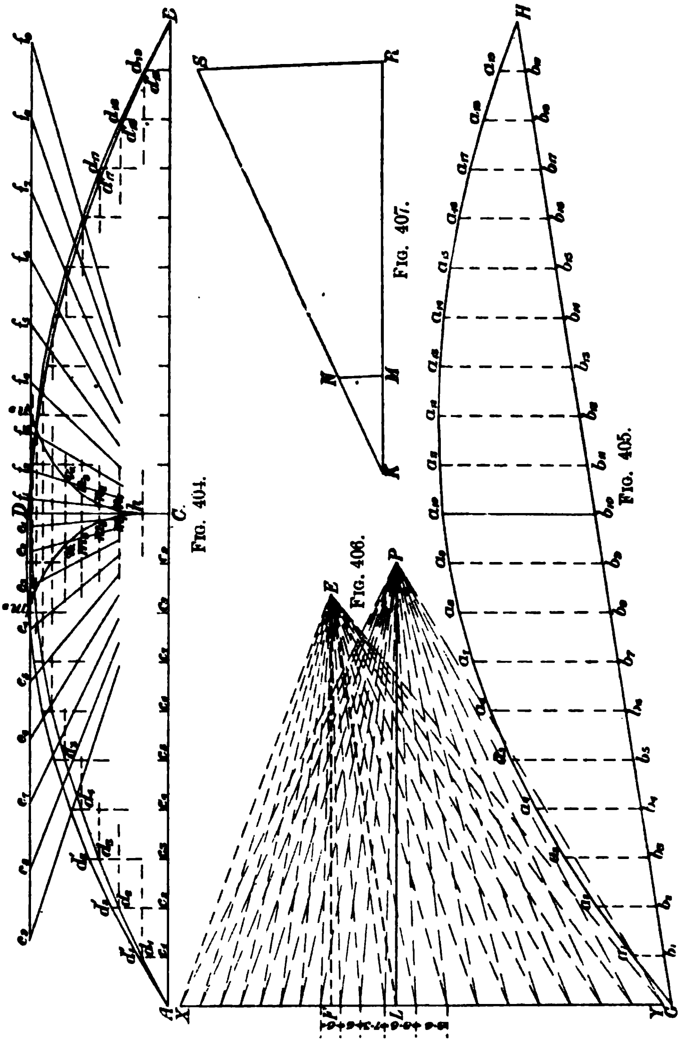


FIG. 404.

FIG. 405.

FIG. 406.

FIG. 407.

line of the polygon GH, Fig. 405. Now, since the extremities of the neutral axis of the rib A and B remain unchanged in position at the springing, whatever loads, producing deflections, act upon the rib between these points, it follows that the sum of all these deflections, both horizontally and vertically, between these points A and B must equal zero; hence—

$$\sum_{B}^{A}\frac{n\mathrm{M}x}{\mathrm{EI}} = \sum_{B}^{A}\frac{n\mathrm{M}y}{\mathrm{EI}} = 0$$

$$\therefore \sum_{B}^{A}\mathrm{M}x = \sum_{B}^{A}\mathrm{M}y = 0.$$

In consequence of the hinges, the ends of the rib may rise or fall vertically without affecting the conditions of bending, so that it is only necessary to consider the sum of the horizontal deflections—

$$\sum_{B}^{A}\mathrm{M}y = 0.$$

But since the neutral axis of the rib $\mathrm{A}d_1d_2d_3$ . . . B may be considered as an equilibrium polygon for its proper load, denote its ordinates by M_d, and the ordinates of the equilibrium polygon $\mathrm{G}a_1a_2a_3$ . . . H by M_a. Then we have seen that—

$$\mathrm{M} = -\mathrm{H}(\mathrm{V}' - \mathrm{V})$$

$$\therefore \sum_{B}^{A}\mathrm{M}_a y - \sum_{B}^{A}\mathrm{M}_d y = \sum_{B}^{A}\mathrm{M}y = 0$$

$$\sum_{B}^{A}\mathrm{M}_a y = \sum_{B}^{A}\mathrm{M}_d y.$$

To draw the horizontal deflection polygon for the arched rib subjected to the bending moments denoted by M_a, we may treat it as a vertical beam fixed at C and loaded at points at a distance from C equal to the vertical heights of the points d_1d_2 . . . above AB, using, however, the ordinates M_a in the same manner as loads in an ordinary equilibrium polygon.

Assume C as the pole, and set off the ordinates M_a along a line $\mathrm{D}f_9$, parallel to AB, to a convenient scale, thus—make

$$\mathrm{D}f_1 = \frac{a_1b_1 + a_{19}b_{19}}{4},\ f_1f_2 = \frac{a_2b_2 + a_{18}b_{18}}{4},$$

and so on.

Draw the lines from $f_1f_2f_3$ . . . f_9 to C, also the lines from $d_1d_2d_3$, etc., parallel to AB cutting CD. The intersection of the line from d_1d_{19} with CD will be taken as one point in the deflection polygon, since the moments at A and B are zero. Draw hn_1 parallel to $\mathrm{C}f_1$ intersecting d_2d_{18} in n_1, and n_1n_2 parallel to $\mathrm{C}f_2$ intersecting d_3d_{17} in n_2, and so on, thus obtaining the deflection polygon hn_1n_2 . . . n_9. In a similar manner we obtain the deflection polygon hm_1m_2 . . . m_9, using the ordinates M_d instead of M_a and setting them off on the left of CD. The ordinates have been taken in pairs in each case, since each member of the pair acts at the same distance y from C. Since Dn_9 represents the maximum deflection for bending moments M_a, and Dm_9 for M_d, it follows that, since $\sum\limits_{B}^{A} M_a y = \sum\limits_{B}^{A} M_d y$, Dn_9 should equal Dm_9; but Dm_9 is less than Dn_9, hence the ordinates M_a are too large or the polar distance EF is too small, since by Culman's principle the ordinates multiplied by the polar distance equal the bending moment. The moments represented by the polygon M_a must therefore be decreased by multiplying them by the ratio $\frac{Dm_9}{Dn_9}$. Set off a line KM $= Dm_9$, Fig. 407, and draw MN at right angles to KM. With K as centre and Dn_9 as radius, cut MN in N. Then the polar distance must be increased in the ratio KN to KM. Make KR equal to the assumed polar distance, and draw RS parallel to MN; then KS is the true polar distance for the equilibrium polygon M_a. To find the true pole, draw EL parallel to GH, then draw LP parallel to EF and equal to KS. The vertical reactions are LX and LY at B and A, or 63·5 and 75·2 tons respectively. We may find these reactions independently by calculation. The horizontal thrust found by measuring LP is 132·5 tons. The horizontal thrust may be calculated by the following formula, which is due to Prof. Winkler :—

Let H = the horizontal thrust.
S = the span, in this case 147 ft.
W = a load concentrated at any point of the rib.
x = horizontal distance of the load W from the crown of the rib.
r = the rise of the rib = 20 ft.

Then—

$$H = \frac{5W(5s^2 - 4x^2)(s^2 - 4x^2)}{128rs^3}.$$

The only variable quantities are W and x, therefore substitute for W the loads concentrated at the panel-points of the rib, and put $x = 0, \frac{s}{20}, \frac{2s}{20}, \frac{3s}{20}$, etc.; thus let $x = \frac{s}{20}$, and $W = 8·6$;

$$H = \frac{5 \times 8·6 \times 147 \{[5 - 4(\frac{1}{20})^2][1 - 4(\frac{1}{20})^2]\}}{128 \times 20} = 12·19 \text{ tons}.$$

Repeating this calculation for each of the loads and adding the results together, we obtain for the horizontal thrust 134 tons, which checks very well with the value 132·5 tons found by graphical construction.

Having obtained the horizontal thrust, we may find the stresses in the arched rib by the method of moments, or we may join the pole P with the points of the force polygon XLY, and construct the equilibrium polygon $Ad_1'd_2'B$. The intercepts between the polygon last drawn and the rib, multiplied by the polar distance or horizontal thrust LP, equal the bending moments at the points d_1, d_2, d_3, etc. The line Ad_1d_2B denotes the neutral axis of the arched rib, and may be considered as the line passing through the centres of gravity of the cross-section of the rib.

The depth of the rib should be made from $\frac{1}{40}$ to $\frac{1}{50}$ of the span in order to give sufficient stiffness under partial loading; in the example selected the depth is made 3 ft. 6 in. between the centres of gravity of the top and bottom flanges. The system of bracing adopted between the flanges is sufficiently illustrated in Fig. 408.

The stresses in the various members of the arch are shown by means of the stress diagram, Fig. 409, which needs no explanation. As a check on the accuracy of the work, a few of the members should be checked by moments; thus $c - 37$, by measurement from the stress diagrams, is found to be 86·5 tons; by moments it is 87 tons. Bar 21-22 is subjected to a compressive stress of 12·5 tons, $d - 35$ to a compressive stress of 93·5 tons, $z - 10$ to a compressive stress of 127 tons.

The foregoing method illustrates how the stresses in an arched rib hinged at the springing may be determined. In any actual case it is necessary to determine the horizontal thrust and the various stresses produced by it for the various positions of the live load. The stresses should then be tabulated and the rib designed for the maximum stress in the various bars.

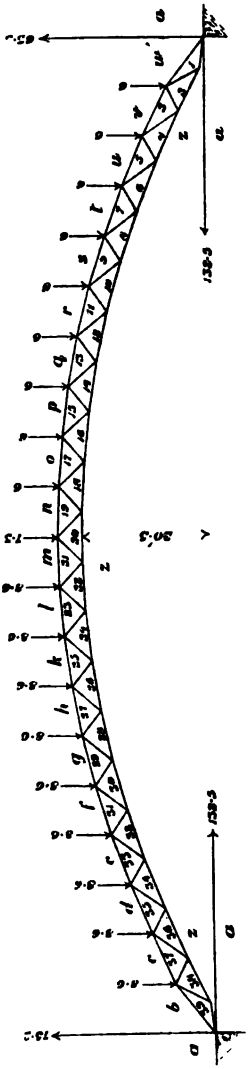


Fig. 408.

In consequence of the continuity of the rib at the crown, it will be subject to stresses, in addition to the foregoing, from changes in temperature. Since the ends of the rib are hinged, any variation in temperature will cause the length to vary and the crown to rise or fall. The effect of a rise in temperature above the mean temperature for which there is no stress is to cause a horizontal thrust at the ends.

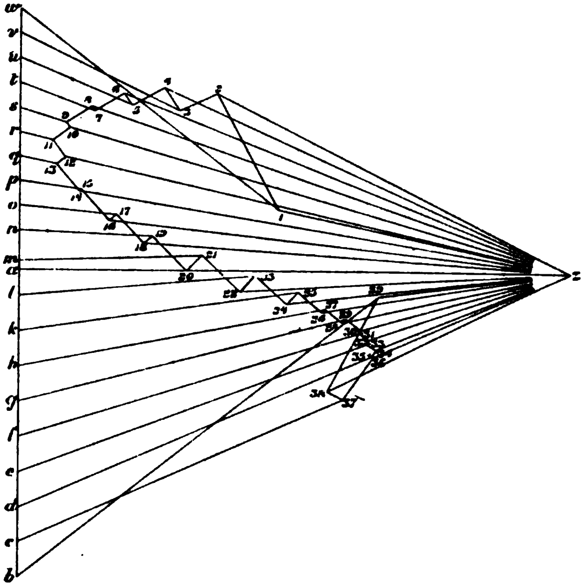


FIG. 409.

The deflection in the centre of the rib caused by a variation of temperature of ± 50 Fahr. will be 1 in. approximately, or $\frac{1}{12}$ of a foot. The following approximate formulæ may be used to calculate the deflection of a parabolic arched rib due to its expansion or contraction from variations of temperature :—

Let s = the length of half the arched rib.
r = the rise, or central deflection.
y = the half-span.

Then—

$$s = \sqrt{y^2 + \tfrac{4}{3}r^2}.$$

To find the deflection from the length of the curve and the span, let r' = the deflection and s_1 = the altered length ; then—

$$r' = \sqrt{(\tfrac{3}{4}s_1^2 - y^2)}.$$

The horizontal thrust due to this deflection may be considered as acting along the line AB.

Comparing the following equations—

$$\Sigma Wx = Hy$$
$$\Sigma nMx = \Sigma M'x = EIV$$

where H = the horizontal thrust required.

$$M' = nM, \text{ and } M' = nHy, \text{ also } H = \frac{EIV}{\Sigma nyx}.$$

From these equations it is clear that if M′ be taken as vertical loading, and EI as the polar distance, the ordinates of the corresponding equilibrium polygon will represent the deflections to the same scale as x; also that M′ and EI are of the same denomination, and hence must be measured to the same scale. It is necessary now to consider the value EI, to do which we must assume a cross-section which can be easily designed approximately by means of the stresses obtained disregarding temperature. The section should be dimensioned in feet for finding the moment of inertia, and E may be taken as 26,000,000 lbs. per sq. in., or 1,670,400 tons per sq. ft. It will then be found that the product EI is about 1,600,000 foot tons with the assumed section of rib shown in Fig. 410.

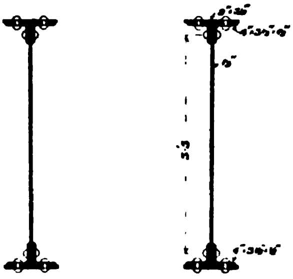


FIG. 410.

In Fig. 411 draw LP parallel to AB, and equal to EI, or 1,600,000 foot tons to scale. If LP is made 16 in., then 1 in. will represent 100,000 foot tons.

Set off the ordinates a_1b_1, a_2b_2, a_3b_3, etc., ... DC on the load line LM, making each one-half of its length in the polygon Aa_1a_2 . . . B, in order to avoid an unnecessarily large figure. Draw the polygon Fc_1c_2 . . . E, with P as the pole representing the actual deflections of the rib to an exaggerated scale; thus LF represents the deflection of the point D, which we have assumed to be 1 in., which will measure 3·1 in. if the rib is drawn to a scale of 8 ft. to an inch, corresponding with 8 × 3·1 = 24·8 ft.; the bending moments must be reduced in the ratio of 24·8 to $\frac{1}{12}$ or $\frac{1}{297\cdot6}$.

The bending moment at the point D is $\dfrac{\frac{2\cdot56}{2}\times 100000}{297\cdot6}$, and the horizontal thrust is $H = \dfrac{M_1}{ny} = \dfrac{1\cdot28 \times 100000}{297\cdot6 \times 7\cdot35 \times 20\cdot5} = 2\cdot8$ tons.

Since $H = \dfrac{EIV}{\Sigma nyx}$, we might calculate the horizontal thrust by this equation, but it would be necessary to measure the quantities yx and take their sum.

Professor Winkler's formula is as follows:—

$$H = \frac{15EIaet}{8ar^2 + 15I}$$

where a = the area of the rib = 48 sq. in.
e = the coefficient of expansion = 0·0000068.
t = the temperature = 50°.

$$\therefore H = \frac{15 \times 1600000 \times \frac{48}{144} \times 0{\cdot}0000068 \times 50}{8 \times \frac{48}{144} \times 20{\cdot}5^2 + 15 \times 0{\cdot}9} = 2{\cdot}4 \text{ tons.}$$

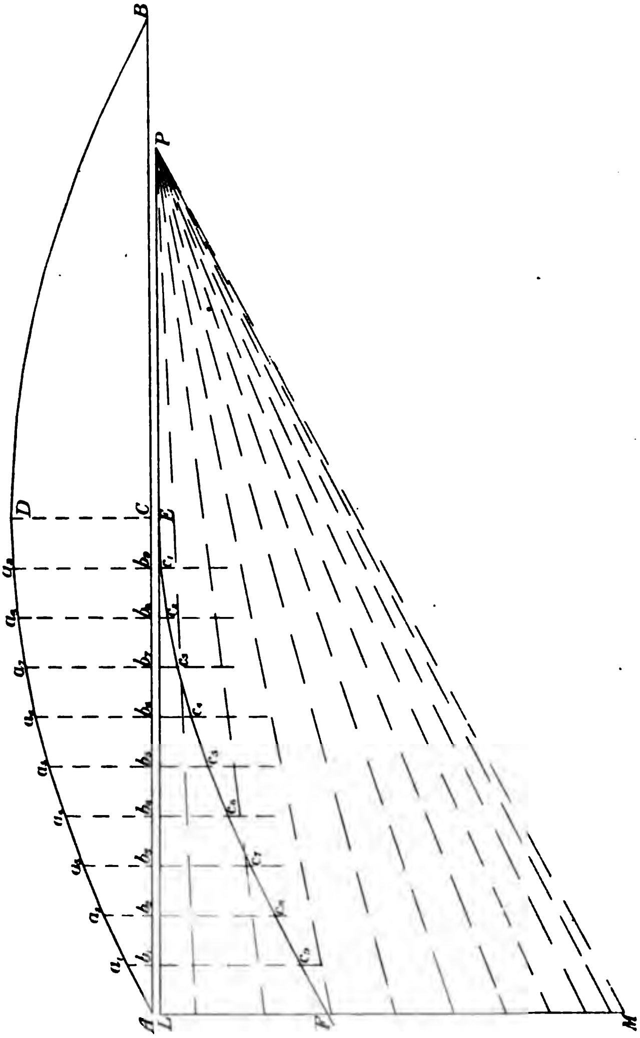


Fig. 411.

This agrees very well with result found by means of the equilibrium polygon.

In the Harlem bridge the range of temperature allowed was

75° Fahr. above and below the mean, and the increase in the stresses amounted to 1 ton per sq. in.

Arched Rib with Fixed Ends.—The equilibrium polygon for an arched rib with fixed ends is not a closed one, and it corresponds with that obtained from a system of forces whose resultant is a couple.

Let W_1, W_2, . . . W_8, Fig. 412, denote a series of vertical forces acting at points of a horizontal line AB.

Let $W_1 + W_2 + W_3 + W_7 + W_8 = W_4 + W_5 + W_6$.

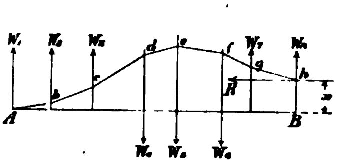


FIG. 412.

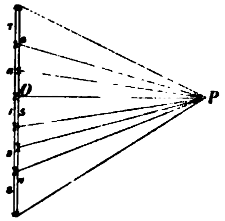


FIG. 413.

Set off the forces on a double line, Fig. 413; draw a horizontal line through O, and on it take any point P as a pole, so that the line PO represents the polar distance. Join P with the angular points of the force polygon, and draw the polygon *Abcdefgh*, Fig. 412, the sides of which are parallel to the lines converging at P. This polygon will not close, showing that the resultant is a couple whose moment is the polar distance PO multiplied by the lever arm x, or Rx, Fig. 412.

In the rib with fixed ends, as with free ends, we have—

$$M = -H(V' - V),$$

for to fix the rib at the ends we have only to apply the proper couple, whose moment is m, remembering that the vertical dimensions of the equilibrium polygon will be increased over the free-ended rib by a constant quantity, which is H.

Take moments about any point of the neutral axis of rib, and there results—

$$M = \Sigma Wx + m - H\left(V' + \frac{m}{H}\right) = \Sigma Wx - HV'$$

and since $\Sigma Wx = HV$, $M = -H(V' - V)$.

Let ADB, Fig. 414, denote the neutral axis of an arched rib fixed at ends of the same span, as the one last considered, and subjected to the same loading. The rib is supposed to be divided into twenty equal parts, each equal to 7·35 ft., which are the panel lengths. As before, we assume that n is equal to the horizontal distance between the panels approximately.

Since the ends of the rib are fixed, the sum of the strains at any given distance from the neutral axis of the rib considered as a girder, and extending between the points A and B, must equal zero.

$$\sum_{B}^{A} na = \sum_{B}^{A} \frac{nM}{EI} = \frac{n}{EI} \sum_{B}^{A} M = 0$$

$$\therefore \sum_{B}^{A} M = 0,$$

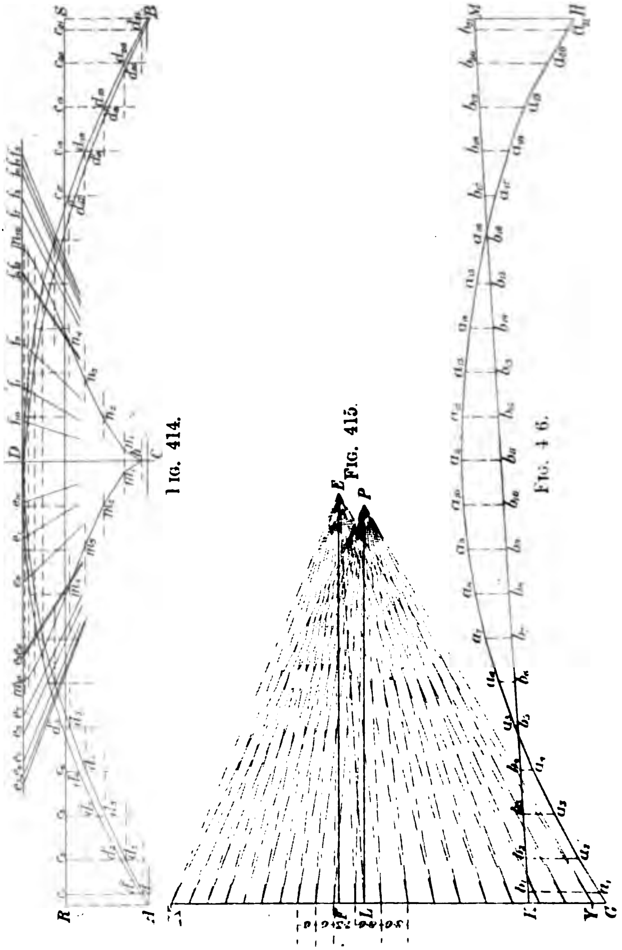


FIG. 414.

FIG. 415.

FIG. 416.

but M = H(V′ − V), where V denotes the ordinates of the polygon and V′ those of the rib.

The above equation therefore shows that the sum of the intercepts (V′ − V) on one side of the rib is equal to the sum on the other.

The sum of the horizontal and vertical deflections must equal zero, as in the free-ended rib, so that the three conditions may be expressed thus—

$$\sum_{B}^{A} M = 0.$$

$$\sum_{B}^{A} Mx = 0.$$

$$\sum_{B}^{A} My = 0.$$

Choose E as the assumed pole, and set off the loads on the line XLY, as in the free-ended rib, and draw the polygon $Ga_1a_2 \ldots H$. The line GH is not the closing line, which must be located so that the equations $\Sigma M = 0$ and $\Sigma Mx = 0$ will be satisfied. Let KM be located by trial, then the algebraic sum of its ordinates must equal zero, and the algebraic sum of the products of each ordinate into its abscissa must equal zero. This line can generally be found by one or two trials. In the figure ordinates are drawn through b_1 and b_{21} at distances of ¼ the panel length from A and B, also K and M respectively, representing loads concentrated at the middle of each half-panel at the ends; these ordinates may be added to the others in the summation, if we divide them first by two; thus we have—

$$\sum_{B}^{A} M_a = \frac{a_1b_1}{2} + a_2b_2 + a_3b_3 + \ldots + a_{20}b_{20} + \frac{a_{21}b_{21}}{2} = 0.$$

The closing line RS of the rib, considered as an equilibrium polygon, can be easily located, because it is parallel to AB, and if the first condition, *viz.*, $\sum_{B}^{A} M_d = 0$, the second condition, $\sum_{B}^{A} M_dx$, must equal zero, since the curve is symmetrical with reference to D.

The third condition requires that $\sum_{B}^{A} M_ay = \sum_{B}^{A} M_dy = 0$. Treating M_a and M_d as loads in the rib with free ends applied at distances y from the assumed origin, the ordinates of the equilibrium polygons will represent the horizontal deflections of the rib.

Set off, therefore, $Df_1, f_1f_2 \ldots f_9f_{10}$ on the right of DC equal to the ordinates of the polygon $Ga_1a_2 \ldots a_{21}H$ taken in pairs, and join the points f_1f_2, etc., with the assumed pole C, and draw the deflection polygon $hn_1n_2 \ldots n_{10}$, as in the case of the rib with free ends.

Draw also the deflection polygon $hm_1m_2 \ldots m_{10}$, using the ordinates of the rib as loads in precisely the same manner as before.

It will be found by measurement that Dm_{10} is greater than Dn_{10}, showing that the quantities Ma are too small, or the polar distances too large. Draw EL parallel to the closing line KM, and LP parallel to AB, make the new polar distance $PL = \frac{Dn_{10}}{Dm_{10}} \times EF$, and draw a new set of lines radiating from P, from which the true equilibrium polygon $Ad_1'd_2'$ . . . B is obtained. Measure LP to the same scale as that used for the loads XLY, and it will be found to be 132 tons. This force acts along the line RS. The bending moment at any point of the arched rib equals the intercept between the rib and the true equilibrium polygon multiplied by the horizontal thrust, *viz.*, 132 tons.

Winkler gives the following formula for locating the line RS :—

$$l = \frac{r(as^2 + 24\mathrm{I})}{3as^2}$$

where l = the height above AB, the other quantities having the same significance as in the free-ended rib. Using the values found for the free-ended rib, we have—

$$l = \frac{20{\cdot}5\{\frac{48}{144} \times \overline{147^2} + 24(0{\cdot}9)\}}{3 \times \frac{48}{144} \times \overline{147^2}} = 6{\cdot}8 \text{ ft.}$$

It scales 6·9 ft. on the diagram.

The reactions at A and B are not the same as in the free-ended rib, being less at A and greater at B; for the fixed rib we measure the segments of the load line LY and LX, Fig. 415, but in the free-ended rib we measure the segments formed by drawing a line from E parallel to the line joining GH.

The temperature stresses in the rib fixed at each end are much greater than in the free-ended rib, and the increase in the horizontal thrust due to the maximum range of temperature may be found by either of the methods explained for the free-ended rib.

By Winkler's formula we have—

$$\mathrm{H} = \frac{45\mathrm{EI}aet}{4ar^2 + 45\mathrm{I}}.$$

Inserting the quantities used in the free-ended rib, we obtain—

$$\mathrm{H} = \frac{45 \times 1600000 \times \frac{48}{144} \times 0{\cdot}0000068 \times 50^\circ}{4 \times \frac{48}{144} \times \overline{20{\cdot}5^2} + 45(0{\cdot}9)} = 14{\cdot}5 \text{ tons.}$$

This increase is due to ± 50°, or 50° on either side of the mean temperature at which there is no stress.

Having found the horizontal thrust for the various positions of the live load, the stresses in the various members of the rib should be found either by moments or by means of a diagram as in Figs. 408 and 409. The stresses obtained should then be tabulated, and the maximum stress on each member obtained.

ANALYTICAL TREATMENT OF ARCHES.

Three-hinged Arched Rib.—Let ABC, Fig. 417, denote the neutral axis of an arched rib hinged at A, B and C, and let W denote a load applied to the rib at a distance of kl from A; also let l denote the span, c the half-span, and r the rise in the centre. The load W

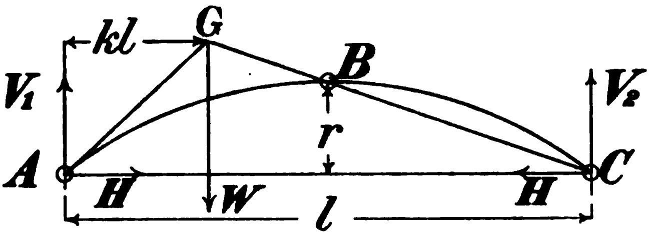


FIG. 417.

produces an inclined reaction at C, which must pass through the hinge at B. Produce CB to meet the vertical through W at G and join GA. Then GA and GC are the lines of pressure R_1 and R_2 respectively. To find the magnitude of R_1 and R_2 and their horizontal and vertical components, set off W, Fig. 418, equal to a load unity acting at G, Fig. 417, and draw lines parallel to CG and GA, also a line H parallel to AC; then—

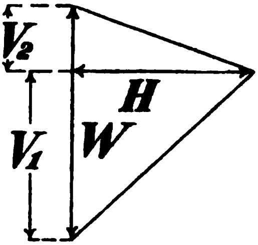


FIG. 418.

$$V_1 + V_2 = 1.$$

Take moments about C—

$$V_1 l = (1 - k)l.$$

Take moments about B—

$$\frac{V_1 l}{2} - Hr - (\tfrac{1}{2} - k)\,l = 0$$

$$\therefore V_1 = 1 - k;\ V_2 = k;\ H = \frac{kc}{r} = \frac{kl}{2r}.$$

If W is situated on the right of B, then—

$$H = (1 - k)\frac{c}{r}.$$

The influence lines for V_1 and H are shown in Figs. 419 and 420.

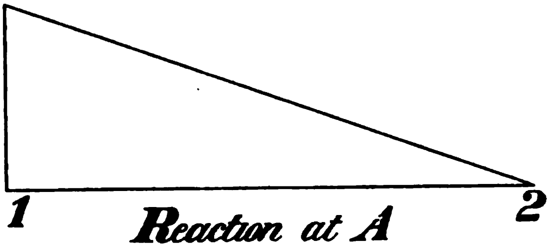


FIG. 419.

The actual values equal the ordinate under the load multiplied by W.

The same reasoning applies to the loads on the right half of the rib. Having found the horizontal and vertical reactions, the values of

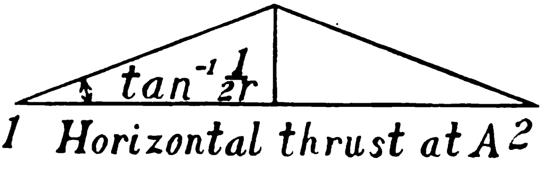


FIG. 420.

which depend merely upon l and r for unity load, the stresses in the various numbers of the arch may be determined by the method of moments or by means of a stress diagram. Variations of temperature need not be considered, as the crown of the arch can rise or fall freely in consequence of the hinged connections.

Braced Arch Hinged at Crown and Springing.—Fig. 421 shows an outline diagram of an arched rib hinged at A, B and C. The span is 160 ft. and the rise 20 ft.; also the number of panels is twenty, so that the panel length is 8 ft. The depth of the rib at A and B is

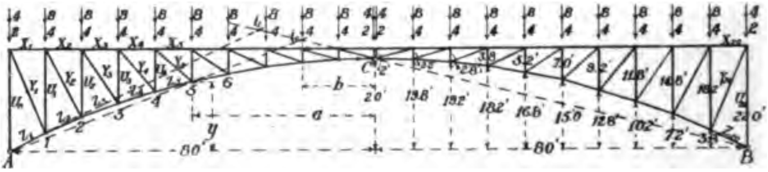


FIG. 421.

22 ft. and at C 2 ft. The figures on the right half give the lengths of the vertical members. The curve ACB is assumed to be a parabola. The dead load at each panel point is taken at eight tons and the live load at four tons.

Stresses in the Horizontal Members X.—Consider the member X_5, cut by a vertical section, which also cuts Y_5 and Z_5, then the moment point for the stress in X_5 is 5. Join A5 and produce it to meet a line drawn through B and C produced at i_5. If any vertical load be placed so that its line of action passes through i_5, it will not produce a stress in X_5, because the lines of action of the reactions R_1 and R_2 coincide with Ai_5 and Bi_5 respectively, and Ai_5 passes through the moment point 5. A load to the right of i_5 causes the reaction to pass below the point 5, and this tends to turn the portion of the arch between 5 and C about the point 5 clockwise. A load to the left of i_5 tends to turn the same portion of the arch about the point 5 contra-clockwise. Hence loads to the right of i_5 produce tension, and loads to the left of i_5 produce compression in the member X_5. Consider the points 9 to 19 loaded with the live and dead load, and the points 1 to 7 loaded with the dead load only. The load at 8 is exactly under i_5 and need not therefore be considered.

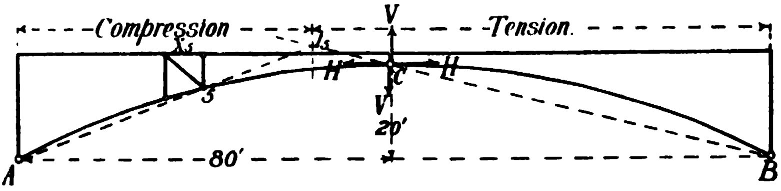


FIG. 422.

Moments about B, Figs. 421 and 422—

$$80V + 20H - 12(\tfrac{80}{2} + 72 + \ . \ . \ . \ + 8) = 0.$$

Moments about A—

$$80V - 20H + 8(\tfrac{80}{2} + 72 + \ . \ . \ . \ + 8) + 4(\tfrac{80}{2} + 72) = 0$$

$$\therefore \ H = 211{\cdot}2 \text{ and } V = 7{\cdot}2.$$

Moments about the point 5—

$$0 = -7X_5 - 211{\cdot}2 \times 5 + 7{\cdot}2 \times 40 + 4(20 + 32) + 8(20 + 32 + \ . \ . \ . \ + 8)$$

$$\therefore \ X_5 = 34{\cdot}3 \text{ nearly.}$$

Consider the points 1 to 7 loaded with the live and dead load, and the points 9 to 19 loaded with the dead load only, the load at 8 being neglected, as it is under i_5 as before.

Moments about B—

$$0 = -80V + 20H - 8(\tfrac{80}{2} + 72 + \ . \ . \ . \ + 8).$$

Moments about A—

$$0 = -80V - 20H + 8(\tfrac{80}{2} + 72 + \ . \ . \ . \ + 8) + 4(56 + \ . \ . \ . \ + 8)$$

$$\therefore \ H = 182{\cdot}4 \text{ and } V = 5{\cdot}6.$$

Moments about the point 5—

$$0 = -7X_5 - 182{\cdot}4 \times 5 - 5{\cdot}6 \times 40 + 4(8 + 16) + 8(20 + 32 + \ . \ . \ . \ + 8)$$

$$\therefore \ X_5 = -\ 34{\cdot}3 \text{ nearly.}$$

The stress in the member X_5 is ± 34·3, so that under the full load acting at every panel point there is no stress in the bar X_5. It follows, therefore, that the dead load will not produce a stress in X_5, or in any of the members X_1 to X_{20}, if the curve ACB is a parabola. The dead load need not therefore be considered for these members. To find the stress in the member X_1: The point i_1 lies on the line BC produced, and also on the line A1 produced, and the vertical line through i_1 cuts the panel 6-7 24·8 ft. to the left of the point C, so that the points 7 to 19 must be loaded to produce the maximum tension, and the points 1 to 6 for the maximum compression. Proceeding as before, using only the live loads, we have—

$$0 = 80\text{V} + 20\text{H} - 4(\tfrac{80}{2} + 72 + \ldots + 8)$$
$$0 = 80\text{V} - 20\text{H} + 4(\tfrac{80}{2} + 72 + 64 + 56)$$
$$\therefore \text{H} = 63{\cdot}2 \text{ and } \text{V} = 4{\cdot}2.$$

Also taking moments about the point 1 we have—

$$0 = -18{\cdot}2\text{X}_1 - 63{\cdot}2 \times 16{\cdot}2 + 4{\cdot}2 \times 7{\cdot}2 + 4(\tfrac{72}{2} + 64 + 56 + 48)$$
$$\therefore \text{X}_1 = 5{\cdot}2 \text{ nearly.}$$

Again—

$$0 = -80\text{V} + 20\text{H}$$
$$0 = -80\text{V} - 20\text{H} + 4(48 + \ldots + 8)$$
$$\therefore \text{H} = 16{\cdot}8 \text{ and } \text{V} = 4{\cdot}2.$$

Also—

$$0 = -18{\cdot}2\text{X}_1 - 16{\cdot}8 \times 16{\cdot}2 - 4{\cdot}2 \times 72 + 4(40 + \ldots + 8)$$
$$\therefore \text{X}_1 = -5{\cdot}2 \text{ nearly.}$$

In a similar manner the remaining stresses in the members X may be determined. If we denote the distance of the vertical line drawn through any of the points i by b, and a the distance of the moment point, each measured from the central hinge, then it can be proved that the vertical line through i, or the load boundary, is given by the following equation :—

$$b = \frac{al}{2(l + a)}.$$

For the member X_5 for instance—

$$b = \frac{40 \times 160}{2(160 + 40)} = 16 \text{ ft., or over the point 8.}$$

For the member X_1—

$$b = \frac{72 \times 160}{2(160 + 72)} = 24{\cdot}8 \text{ about.}$$

This equation may be proved in the following manner: Let V_1 and H denote the vertical and horizontal reactions at A, Fig. 421, and y the ordinate at the moment point for the member under consideration. Then since any load at i produces no stress in the X member, we have—

$$\text{V}_1\left(\frac{l}{2} - a\right) = \text{H}y,$$

$$\text{but } \text{V}_1 = \frac{l + 2b}{2l}\text{W} \text{ and } \text{H} = \frac{l - 2b}{4r}\text{W}$$

$$\therefore y = \frac{(l + 2b)(l - 2a)r}{l(l - 2b)}.$$

But since the curve ACB is a parabola—

$$\frac{r-y}{r} = \frac{4a^2}{l^2} \text{ and } y = r\left(1 - \frac{4a^2}{l^2}\right)$$

$$\therefore \frac{l+2b}{l-2b} = \frac{l+2a}{l} \text{ and } b = \frac{al}{2(l+a)}.$$

The load boundary may be determined by means of this equation or by means of the construction already explained. The equation will be used in connection with the stresses in a three-hinged arch bridge shown in Plate IV., but in this case l denotes the half-span, and—

$$b = \frac{al}{2l+a}.$$

This load boundary only applies to the X-bars in Fig. 421.

Stresses in the Diagonals.—Member Y_5. The moment point e_5 is found at the join of Z_5 with X_5, at a distance of 14·56 ft. from C, and the lever arm of y_5 is 22 ft. The load boundary is found by drawing a vertical through the join of Ae_5 and BC produced to meet in i_5 between 8 and 9 (Figs. 421 and 423). Loads at 1 to 8 produce the maximum tension and loads 9 to 19 the maximum compression in Y_5. Loads 1 to 8—

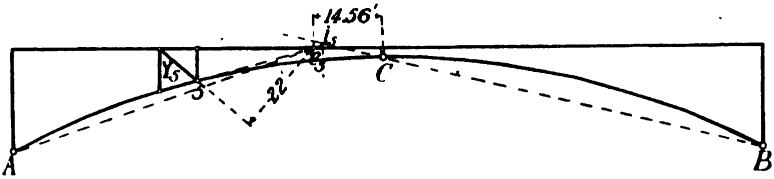


FIG. 423.

$$0 = 20\text{H} - 80\text{V}$$

$$0 = -20\text{H} - 80\text{V} + 4(8 + \ldots + 64)$$

$$\therefore \text{H} = 28{\cdot}8;\ \text{V} = 7{\cdot}2$$

$$\therefore 22\text{Y}_5 - 7{\cdot}2 \times 14{\cdot}56 + 28{\cdot}8 \times 2 - 4(1{\cdot}44 + 9{\cdot}44 + 17{\cdot}44 + 25{\cdot}44) = 0$$

$$\therefore \text{Y}_5 = 11{\cdot}9.$$

Loads at 9 to 19—

$$0 = 20\text{H} + 80\text{V} - 4(\tfrac{80}{2} + 72 + \ldots + 8)$$

$$0 = -20\text{H} + 80\text{V} - 4(\tfrac{80}{2} + 72)$$

$$\therefore \text{H} = 51{\cdot}2;\ \text{V} = 7{\cdot}2$$

$$\therefore 22\text{Y}_5 + 7{\cdot}2 \times 14{\cdot}56 + 51{\cdot}2 \times 2 + 4(7{\cdot}28 + 6{\cdot}56) = 0$$

$$\therefore \text{Y}_5 = -11{\cdot}9.$$

Complete loading, therefore, will not produce any stress in the bar Y_5, and the maximum tensile stress is equal to the maximum compressive stress. In the remaining diagonal bars it will not be necessary to consider either the dead load or the loading to produce the maximum compressive stress. In the bar Y_8 the point e_8 falls to the right of the point C, but i_8 falls on the left of C, therefore i_8 defines the load boundary as before; but there is another load boundary between the points 7 and 8. The loads at the panel points 8 and 9 produce the maximum tension, and the remaining loads the maximum compression in Y_8. Considering the loads at 8 and 9, we obtain: $H = 13{\cdot}6$ and $V = 3{\cdot}4$. The lever arm for the stress in Y_8 is 10·1 ft., and the point e_8 is 6·4 ft. to the right of the point C; therefore—

$$0 = 10{\cdot}1Y_8 + 13{\cdot}6 \times 2 + 3{\cdot}4 \times 6{\cdot}4 - 4(14{\cdot}4 + 22{\cdot}4)$$

$$\therefore\ Y_8 = 9{\cdot}7.$$

In a similar manner it may be shown that the maximum compression is—

$$Y_8 = -9{\cdot}7.$$

For the members Y_9 and Y_{10} the moment point is not only on the right of the point C, but i_9 and i_{10} both fall below the point C on the right, so that they no longer define a load boundary for Y_9 and Y_{10}, as loads on the right half of the arch do not produce a reaction passing through the points B and C; but there is a load boundary in the panel cut by the section, between 8 and 9 for Y_9, and between 9 and 10 for Y_{10}.

Stresses in the Vertical Members.—For the vertical member U_4 the moment point is the same as for the diagonal Y_5, or 14·56 to the left of the point C, and the panel points 1 to 8 must be loaded to produce the maximum compression, and the panel points 9 to 19 to produce the maximum tension in the member U_4. $H = 28{\cdot}8$ and $V = 7{\cdot}2$. The lever arm for the stress in U_4 is 33·44, and the stress when the points 1 to 8 are loaded is—

$$0 = -33{\cdot}44\,U_4 - 7{\cdot}2 \times 14{\cdot}56 + 28{\cdot}8 \times 2 - 4(1{\cdot}44 + 9{\cdot}44 + 17{\cdot}44 + 25{\cdot}44 + 33{\cdot}44)$$

$$\therefore\ U_4 = -11{\cdot}8.$$

If the panel points 9 to 19 are loaded for the maximum tension we obtain: $H = 51{\cdot}2$ and $V = 7{\cdot}2$; therefore—

$$0 = -33{\cdot}44\,U_4 + 51{\cdot}2 \times 2 + 7{\cdot}2 \times 14{\cdot}56 + 4(7{\cdot}28 + 6{\cdot}56)$$

$$\therefore\ U_4 = 7{\cdot}8.$$

For complete loading with the live load the stress is—

$$-11{\cdot}8 + 7{\cdot}8 = -4,$$

or the same as the panel point load. The dead load stress is therefore -8. The maximum compressive stress in U_4 is—

$$-11{\cdot}8 - 8 = -19{\cdot}8.$$

There is no tensile stress in U_4.

For the member U_7 the moment point is the same as for Y_8, and the panel points 7, 8 and 9 must be loaded for the maximum compressive stress, and the panel points 1 to 6 and 10 to 19 for the maximum tensile stress. In the former case $H = 19{\cdot}2$ and $V = 4{\cdot}8$. The lever arm for the stress in U_7 is 30·4 ft.—

$$0 = -30{\cdot}4\,U_7 + 19{\cdot}2 \times 2 + 4{\cdot}8 \times 6{\cdot}4 - 4(14{\cdot}4 + 22{\cdot}4 + 30{\cdot}4)$$
$$\therefore\ U_7 = -6{\cdot}5.$$

For the points 1 to 6 and 10 to 19 loaded we have: $H = 56{\cdot}8$ and $V = 5{\cdot}8$—

$$0 = -30{\cdot}4\,U_7 + 56{\cdot}8 \times 2 - 5{\cdot}8 \times 6{\cdot}4$$
$$\therefore\ U_7 = 2{\cdot}5.$$

The full live load produces a stress—

$$-6{\cdot}5 + 2{\cdot}5 = -4.$$

As in the last case the maximum compressive stress in U_7 is—

$$-8 - 6{\cdot}5 = -14{\cdot}5.$$

In a similar manner the remaining stresses in the verticals can be determined.

Stresses in the Lower Members.—Consider the member Z_5; the moment point is immediately over the point 4, and the lever arm is 8·4 ft. The load boundary is between the points 5 and 6, so that the panel points 1 to 5 must be loaded to produce the maximum tension and 6 to 19 for the maximum compression. In the former case, $H = 12$ and $V = 3$; also the stress in Z_5 is—

$$0 = 8{\cdot}4Z_5 - 3 \times 48 + 12 \times 2 + 4 \times 8.$$
$$\therefore\ Z_5 = 10.$$

For the points 6 to 19 loaded, $H = 68$ and $V = 3$; also—

$$0 = 8{\cdot}4Z_5 + 3 \times 48 + 68 \times 2 + 4(24 + 40 + \ldots + 16)$$
$$\therefore\ Z_5 = -98.$$

For the full live load the stress in Z_5 is—

$$10 - 98 = -88.$$

Therefore the dead load stress is -176, and the maximum stress in Z_5 is—

$$-(176 + 98) = -274.$$

The stress in the lower members is clearly the horizontal thrust multiplied by the secant of the angle the member makes with the horizon. The stresses in the remaining bars may be found in a similar manner to that illustrated. It should be observed that the vertical member U_{10} supports only the panel load at the point 10, and also that there is no stress in the member X_{10}.

Arched Rib with Hinges at A and B (Fig. 424).—In this case we cannot find the value of H by pure statics, but—

$$V_1 = (1 - k)W;\ V_2 = kW.$$

Let the point B be supposed free to expand to B_0, a distance horizontally equal to Δ. Suppose also that a horizontal force H applied at B_0 brings the rib back to B. We can determine the value of H by considering that the value Δ due to W is equal to the value of Δ due to H. If we consider the horizontal force applied at B_0 is equal to unity, the external work is—

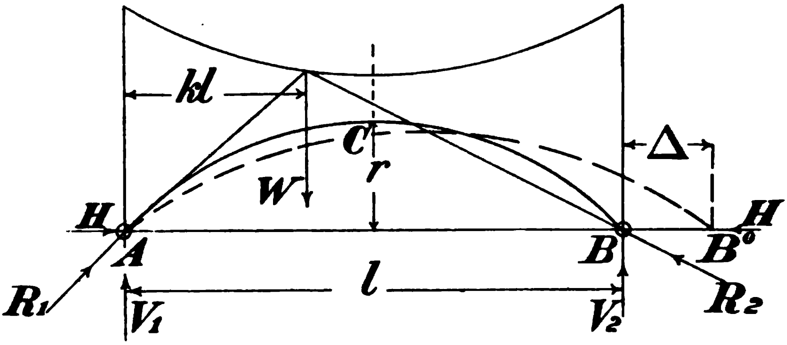


FIG. 424.

$$\frac{\Delta}{2}.$$

The external work equals the internal work due to bending stress. Let *ds* denote an elementary length of the rib, Fig. 425; I, the moment of inertia of the cross-section about its neutral axis; M, the bending moment for the vertical forces; then—

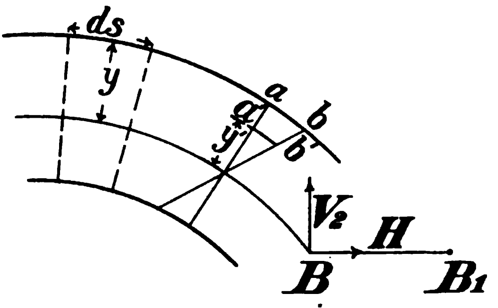


FIG. 425.

$$f = \frac{My}{I}.$$

$$\text{The strain } ab\text{, Fig. 425} = \frac{My}{I}\frac{ds}{E}.$$

$$\text{,, ,, } a'b' \text{ ,, } \frac{My'}{I}\frac{ds}{E}.$$

Let *m* denote the bending moment due to the horizontal force at B equal to unity.

The intensity of stress due to this force is—

$$\frac{my'}{I}.$$

If a denote the area of the fibre, the total stress upon it is—

$$\frac{may'}{\mathrm{I}}.$$

The internal work is—

$$\frac{may'}{\mathrm{I}}\left(\frac{\mathrm{M}y'ds}{2\mathrm{EI}}\right) = \frac{\mathrm{M}may'^2ds}{2\mathrm{I}^2\mathrm{E}},$$

and since $\Sigma ay'^2 = \mathrm{I}$, the internal work of the whole cross-section is—

$$\int\frac{\mathrm{M}mds}{2\mathrm{EI}}$$

$$\therefore \int\frac{\mathrm{M}mds}{2\mathrm{EI}} = \frac{\Delta}{2} \text{ and } \int\frac{\mathrm{M}mds}{\mathrm{EI}} = \Delta.$$

This is the horizontal displacement of B due to W. In a similar manner if M_1 denote the bending moment due to H, or $\mathrm{M}_1 = -\mathrm{H}m$.

$$\text{Then } \Delta = \frac{1}{\mathrm{H}}\int\frac{\mathrm{M}_1{}^2ds}{\mathrm{EI}} = \mathrm{H}\int\frac{m^2ds}{\mathrm{EI}}$$

which is the horizontal displacement of B due to H.

$$\therefore \mathrm{H}\int\frac{m^2ds}{\mathrm{EI}} = \int\frac{\mathrm{M}mds}{\mathrm{EI}}$$

$$\text{and } \mathrm{H} = \frac{\int\frac{\mathrm{M}mds}{\mathrm{EI}}}{\int\frac{m^2ds}{\mathrm{EI}}}$$

Parabolic Arch Rib Hinged at Springing.—If we take the origin at A, the equation of the parabola ACB representing the centre line of the rib is—

$$y = \frac{4r}{l^2}\left(lx - x^2\right) = 4r\left(\frac{x}{l} - \frac{x^2}{l^2}\right).$$

Let a load be applied at d, Fig. 426, then—

$$\mathrm{V}_1 = \mathrm{W}(1 - k);\ \mathrm{V}_2 = \mathrm{W}k.$$

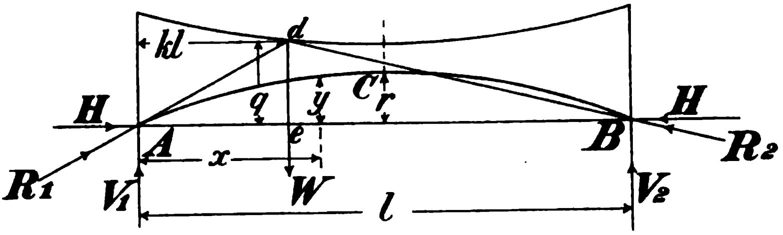


Fig. 426.

The value of H is found in the following manner:—

Since the deformation of the rib is due to the horizontal and vertical reactions producing bending stresses, we have for any section on the left of W—

$$M = V_1x - Hy = W(1 - k)x - Hy.$$

At any section on the right of W—

$$\begin{aligned} M &= V_1x - W(x - kl) - Hy \\ &= W(1 - k)x - W(x - kl) - Hy. \end{aligned}$$

Let M_1 denote the bending moment due to the vertical forces, and M_2 that due to the horizontal thrust H_1, then—

$$M_1 = V_1x, \text{ or } V_1x - W(x - kl)$$

$$M_2 = - Hy.$$

In general, the bending moment due to a horizontal thrust of unity acting away from B is equal to y, and—

$$M = M_1 + M_2 = M_1 + Hy,$$

but since—

$$H = \frac{\int \frac{Mmds}{EI}}{\int \frac{m^2ds}{EI}}$$

and $I = I_c \sec i$, where I_c denotes the moment of inertia at the crown, and i the inclination of the tangent to the curve, also $ds = dx \sec i$, then—

$$H = \frac{\int_o^l W(1 - k)xydx + \int_{kl}^l \{W(1 - k)x - W(x - kl)\}ydx}{\int_o y^2dx}.$$

Substitute for $y = 4r\left(\frac{x}{l} - \frac{x^2}{l^2}\right)$, and integrating we have—

$$H = \frac{5Wl}{8r}(k - 2k^3 + k^4).$$

This is the horizontal thrust for a single load placed on the rib at a distance of kl from A.

Let $S = \frac{l}{2r}$, then—

$$H = \frac{5WS}{4}(k - 2k^3 + k^4).$$

If $W = S =$ unity—

$$H = \tfrac{5}{4}(k - 2k^3 + k^4).$$

The values of H derived from this equation for different values of k are given in the following table:—

TABLE LXXV.

k	0	·01	·02	·03	·04	·05	·06	·07	·08	·09
0	·0000	·0125	·0250	·0370	·0408	·0622	·0745	·0866	·0988	·1107
·1	·1226	·1345	·1461	·1565	·1688	·1797	·1907	·2015	·2119	·2222
·2	·2320	·2418	·2514	·2608	·2697	·2785	·2869	·2949	·3029	·3102
·3	·3175	·3245	·3312	·3375	·3434	·3490	·3542	·3503	·3540	·3680
·4	·3720	·3754	·3787	·3818	·3838	·3857	·3874	·3890	·3899	·3903
·5	·3906	—	—	—	—	—	—	—	—	—

The influence lines for the vertical reaction at A and the horizontal thrust are shown in Figs. 427 and 428.

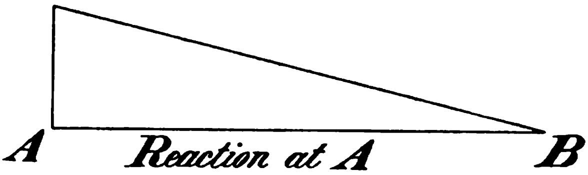


FIG. 427.

Let d be any point where R_1, W and R_2 intersect. As W moves across the span, the point d always lies in a curve, termed the "reaction locus". The ordinate of the point d on this curve is q, and the abscissa kl. In the triangle Aed, Fig. 426, the sides Ae and de are proportional to H and V_1 respectively, or—

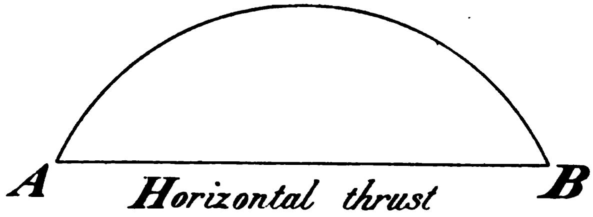


FIG. 428.

$$q = \frac{V_1 kl \times 8r}{5Wl(k - 2k^3 + k^4)} = \frac{8W(1 - k)klr}{5Wl(1 - k)(1 + k - k^2)}$$

$$\therefore q = \frac{8r}{5(1 + k - k^2)} = \frac{1{\cdot}6r}{1 + k - k^2} = \frac{1{\cdot}6r}{1 + k(1 - k)}.$$

The "reaction locus" curve may be plotted from this equation by giving different values to k. The following table gives the value of q when $r = 1$, and when $r = 20$:—

TABLE LXXVI.

k	$1-k$	$k(1-k)$	$1+k(1-k)$	$\frac{1\cdot6}{1+k(-1k)}$	$\frac{1\cdot6\times20}{1+k(1-k)}$
0·05	0·95	0·0475	1·0475	1·527	30·54
0·10	0·90	0·0900	1·0900	1·468	29·36
0·15	0·85	0·1275	1·1275	1·419	28·38
0·20	0·80	0·1600	1·1600	1·379	27·58
0·25	0·75	0·1875	1·1875	1·347	26·94
0·30	0·70	0·2100	1·2100	1·322	26·44
0·35	0·65	0·2275	1·2275	1·303	26·06
0·40	0·60	1·2400	1·2400	1·290	25·80
0·45	0·55	1·2475	1·2475	1·282	25·64
0·50	0·50	1·2500	1·2500	1·280	25·60

For a uniform load equally distributed over the span, we may substitute $W = wd(kl)$—

$$\therefore H = \frac{5wl^2}{8r}\int_0^1 (k - 2k^3 + k^4)dk = \frac{wl^2}{8r}.$$

In this case the equilibrium polygon coincides with the neutral axis of the arch, and there is consequently no bending moment.

If the span of the arch is 160 ft., and the rise 20 ft., the dead and live loads applied at 20 equidistant panel points 8 and 4 tons respectively, we can find the values of the horizontal and vertical reactions in the following manner:—

Add the figures in Table LXXVI. for $k = \cdot05$, $\cdot10$, $\cdot15$, etc., to $\cdot50$, also $\cdot55$ to $\cdot95 = \cdot05$ to $\cdot45$; we obtain the value of H, for $W = S = 1 = 4\cdot989$. Since $W = 8$, and $S = 4$, the value of the horizontal thrust for the dead load is—

$$H = 4\cdot989 \times 4 \times 8 = 159\cdot68 \text{ tons.}$$

For a uniformly distributed load of 1 ton per foot run—

$$H = \frac{160 \times 160}{8 \times 20} = 160 \text{ tons.}$$

The difference 0·352 shows that the equilibrium polygon does not coincide with the neutral axis of the arch, and there is a positive bending moment equal to $0\cdot352y$, where y denotes the ordinate of the neutral axis of the arch. The maximum value of this small moment occurs at the centre, where $y = 20$, and is 0·704 foot tons. The vertical reaction at A is the sum of the 20 ordinates multiplied by 8, Fig. 429.

$$V_1 = \tfrac{1}{20}(1 + 2 + \ldots + 20)8 = 84.$$

For the live load of 4 tons on each of the ten panel points of the right half of the arch—

$$H = 2{\cdot}6898 \times 4 \times 4 = 43{\cdot}0368 \text{ tons.}$$
$$V_2 = \tfrac{1}{20}(10 + 11 + \ldots + 19)4 = 29 \text{ tons.}$$
$$V_1 = \tfrac{1}{20}(1 + \ldots + 10)4 = 11 \text{ tons.}$$

The reaction locus may be determined by plotting the ordinates from Table LXXVI. as in Fig. 429.

The bending moments and stresses for any loading may now be found. The maximum moment at the point 9 occurs when the panel points 6 to 11 are loaded, and when 1 to 5 and 12 to 19 are loaded. The former is positive and the latter negative. The horizontal thrust and the vertical reactions and loads produce compressive and shearing stresses in the arch, which are independent of the depth of the rib, and these must be combined with the stresses due to bending in the manner illustrated for the three-hinged arch, Plates IV. to IV.*b*. In a braced arch the reaction locus must be determined as before and the panel point loads for maximum, positive and negative moments. The stresses in the various members may then be determined as in the case of the three-hinged braced arch, Fig. 421.

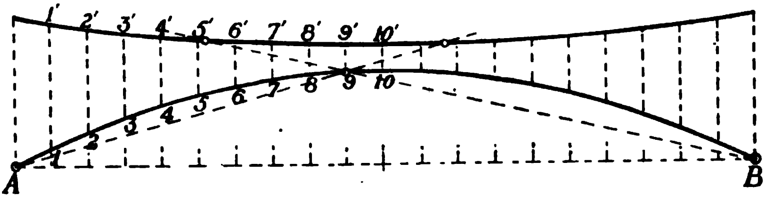


FIG. 429.

Horizontal Thrust producing Shortening of the Arch Rib.—The effect of the horizontal thrust is to shorten the axis of the rib and develop equal and opposite negative reactions. The shortening of an indefinitely small length of the axis ds under compressive stress σ per unit of area is—

$$\lambda = \frac{\sigma ds}{E}.$$

The shortening of the whole arch is—

$$\Delta = \int_0^l \frac{\sigma ds}{E}.$$

σ must be determined from the sectional area, and total stress, in the first place, approximately from a probable cross-section, then—

$$\Delta = \frac{\sigma l}{E}.$$

Making the usual assumptions in regard to I and ds, we have—

$$H = \frac{-\Delta}{\int \frac{m^2 ds}{EI}} = -\frac{\sigma l}{\int \frac{m^2 dx}{I_c}}.$$

where m denotes the bending moment due to $H = 1$ acting outwards at the hinge B, or $m = y$.

Inserting this value for the parabola, and integrating between the limits l and o, we obtain—

$$H = -\frac{\sigma l}{\dfrac{16r^2}{I_c}\displaystyle\int_o^l\left(\frac{x^2}{l^2} - \frac{2x^3}{l^3} + \frac{x^4}{l^4}\right)dx} = -\frac{15\sigma I_c}{8r^2}.$$

This horizontal thrust acts in the opposite direction to that due to the loading, already found. The horizontal thrust due to the dead load is usually a large portion of the total horizontal thrust, and this can be counteracted during erection by wedging up, or otherwise producing initial compression in the rib, so that the span and rise are at mean temperature, exactly equal to the values assumed in the calculations.

Effect of Temperature.—Let e denote the coefficient of expansion and t the rise in temperature, then the increase in the length of the span l is, if one end were free to move—

$$etl.$$

The horizontal thrust developed due to this tendency to lengthen is found thus—

$$etl = H\int\frac{m^2ds}{EI} = \Delta.$$

Since $m = y$, $ds = dx \sec i$, $I = I_c \sec i$, we have—

$$etl = H\int_o^l\frac{y^2dx}{EI_c} = \frac{16Hr^2}{EI_cl^4}\int_o^l(lx - x^2)^2dx$$

$$\therefore\ H = \frac{15EI_cet}{8r^2}.$$

In a similar manner, for a fall in temperature equal to t, we obtain—

$$H = -\frac{15EI_cet}{8r^2}.$$

The vertical reactions V_1 and V_2 are not affected by changes of temperature.

ARCHED RIB FIXED AT THE ENDS.

Here there are no hinges, and the reactions at the ends produce bending moments at the points of fixing, as in the case of fixed beams. The horizontal reaction acts above the springing point, as shown in Fig. 430, over $A = y_1$, and over $B = y_2$. Let W denote the load acting at any point; then, since the load and the horizontal and vertical reactions are in equilibrium, we may equate the moments about $A = 0$—

$$W(c - b) - V_2l - Hy_2 + Hy_1 = 0 \quad . \quad . \quad (1)$$

also—

$$V_1 + V_2 - W = 0 \quad . \quad . \quad . \quad . \quad (2)$$

Let Fig. 431 denote an elementary portion of the arched rib of length ds between two normal sections, ab and cd, in the unloaded rib; also let

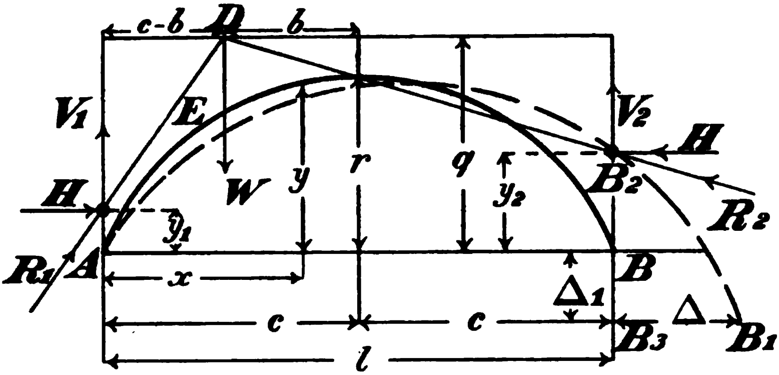


FIG. 430.

$a'b'$ and $c'd'$ denote these sections when the arch is deformed. The inclination of the normals is the same as that of the tangents to the neutral axis at the extremities of the length ds denoted in Fig. 431 by $d\theta$; then—

$$aa' = y_1 d\theta,$$

but aa' is the strain due to the bending moment M producing an extreme fibre stress, $\frac{My_1}{I}$, therefore—

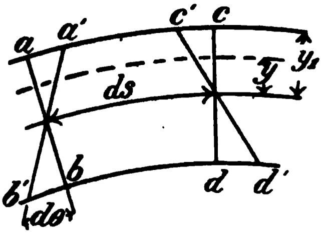


FIG. 431.

$$\frac{\frac{My_1}{I}}{\frac{aa'}{ds}} = \text{E, and } aa' = \frac{My_1}{I}\frac{ds}{E}$$

$$\therefore\ y_1\, d\theta = \frac{My_1 ds}{IE}, \text{ or } d\theta = \frac{Mds}{EI}.$$

If the ends of the rib are fixed the sum of all such changes in the inclination of the tangent must equal zero—

$$\int_0^l \frac{Mds}{EI} = 0 \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

The moments of the horizontal reactions at the ends acting with the leverages y_1 and y_2 respectively must be sufficient to prevent the horizontal and vertical displacements denoted by B_3 B_1 and BB_3, Fig. 430. By reasoning in a similar manner to that used in the two-hinged arch, we can show that the horizontal displacement of the point B may be expressed in terms of the moment M_1 due to V_1, V_2 and W_1, or in terms of M_2 due to H; then if $H = 1$ applied at B producing a bending moment, y, then since the sum of the two displacements must equal zero—

$$\int\frac{M_1 y ds}{EI} + \int\frac{M_2 y ds}{EI} = 0 \quad . \quad . \quad . \quad (4)$$

Again, let $V_2 = 1$ applied at B, the vertical displacement caused by W must be equal and opposite to that caused by H. The moment due to $V_2 = 1$ is denoted by x, then—

$$\int\frac{M_1 x ds}{EI} + \int\frac{M_2 x ds}{EI} = 0 \quad . \quad . \quad . \quad (5)$$

The equations (1) to (5) express the conditions of equilibrium and enable the quantities V_1, V_2, H, y_1 and y_2 to be determined.

PARABOLIC RIB.

Referring to Fig. 430 and taking the point A as origin, we have—

$$y = \frac{4r}{l^2}(lx - x^2).$$

Assume: E constant, $ds = dx \sec i$, $I = I_c \sec i$, $M = M_1 + M_2$, then the controlling conditions are expressed by the following equations:—

$$\int_0^{(c-b)}\frac{M_1 dx}{EI} + \int_{(c-b)}^{l}\frac{M_1 dx}{EI} + \int_0^{l}\frac{M_2 dx}{EI} = 0.$$
$$\int_0^{(c-b)}\frac{M_1 y dx}{EI} + \int_{(c-b)}^{l}\frac{M_1 y dx}{EI} + \int_0^{l}\frac{M_2 y dx}{EI} = 0.$$
$$\int_0^{(c-b)}\frac{M_1 x dx}{EI} + \int_{(c-b)}^{l}\frac{M_1 x dx}{EI} + \int_0^{l}\frac{M_2 x dx}{EI} = 0.$$

The values of M_1 on the left and right of W are respectively—

$$M_1 = V_1 x, \quad M_1 = V_1 x - W\{x - (c - b)\}$$
$$M_2 = - H(y - y_1).$$

Substituting these values of the bending moments in the above equations, and the value of y from the equation to the curve of the rib, we can find y_1, y_2, V_1, V_2, and H. We can also draw the influence lines, and tabulate the results as in the foregoing examples.

The method of influence lines is especially applicable to this case, as there are a few crucial points by which the maximum and minimum sections are determined.

Let $S = \frac{c}{r}$ and $z = \frac{b}{c}$; then—

Influence Line y_1—

$$y_1 = \frac{2}{15}\left(\frac{c - 5b}{c - b}\right)r = \frac{2r}{15}\left(\frac{1 - 5z}{1 - z}\right). \quad \text{(Fig. 432.)}$$

y_1 = the imaginary height of the horizontal thrust above A, Fig. 430.

$$y_2 = \frac{2}{15}\left(\frac{c+5b}{c+b}\right)r = \frac{2r}{15}\left(\frac{1+5z}{1+z}\right);$$

y_2 = the height of H above B.

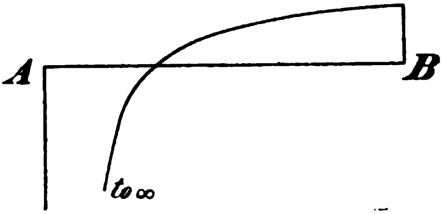


FIG. 432.

Influence Lines Vertical Reactions—

$$V_1 = \tfrac{1}{4}(2-z)(1+z)^2W. \quad \text{(Fig. 433.)}$$
$$V_2 = \tfrac{1}{4}(2+z)(1-z)^2W.$$

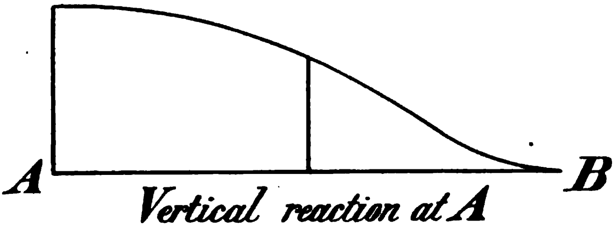


FIG. 433.

Influence Line Horizontal Thrust—

$$H = \frac{15}{32}(1-z^2)^2SW. \quad \text{(Fig. 434.)}$$

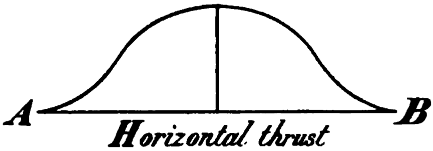


FIG. 434.

TABLE LXXVII.

VERTICAL REACTIONS—V_1.

$$V_1 = \tfrac{1}{4}(2 - z)(1 + z)^2.$$

z	0	·01	·02	·03	·04	·05	·06	·07	·08	·09
0	·5000	·5075	·5150	·5225	·5300	·5375	·5450	·5524	·5599	·5673
·1	·5748	·5822	·5896	·5970	·6043	·6117	·6190	·6263	·6336	·6408
·2	·6480	·6552	·6624	·6695	·6766	·6836	·6906	·6976	·7045	·7114
·3	·7183	·7251	·7318	·7385	·7452	·7518	·7583	·7648	·7713	·7777
·4	·7840	·7903	·7965	·8026	·8087	·8147	·8207	·8265	·8323	·8381
·5	·8438	·8493	·8548	·8603	·8656	·8709	·8761	·8812	·8862	·8911
·6	·8960	·9008	·9054	·9100	·9145	·9188	·9231	·9273	·9314	·9354
·7	·9392	·9430	·9467	·9502	·9537	·9570	·9602	·9634	·9664	·9692
·8	·9720	·9746	·9772	·9796	·9818	·9840	·9860	·9879	·9896	·9913
·9	·9928	·9941	·9953	·9964	·9974	·9982	·9988	·9993	·9997	·9999

TABLE LXXVIII.

HORIZONTAL THRUST—H.

$$H = S \times \frac{15}{32}(1 - z^2)^2.\quad \text{Figures are for } S = \text{unity}.$$

z	0	·01	·02	·03	·04	·05	·06	·07	·08	·09
0	·4687	·4685	·4684	·4680	·4670	·4662	·4652	·4640	·4626	·4610
·1	·4593	·4573	·4551	·4529	·4504	·4478	·4450	·4420	·4388	·4353
·2	·4319	·4281	·4245	·4205	·4163	·4120	·4075	·4027	·3980	·3932
·3	·3882	·3830	·3777	·3721	·3668	·3609	·3550	·3491	·3431	·3370
·4	·3307	·3243	·3179	·3114	·3048	·2981	·2914	·2846	·2777	·2707
·5	·2636	·2566	·2495	·2424	·2352	·2281	·2209	·2136	·2065	·1992
·6	·1920	·1847	·1776	·1704	·1634	·1563	·1493	·1423	·1354	·1287
·7	·1219	·1152	·1092	·1022	·0959	·0897	·0836	·0776	·0718	·0662
·8	·0607	·0554	·0503	·0453	·0406	·0361	·0318	·0277	·0239	·0202
·9	·0169	·0138	·0111	·0085	·0063	·0045	·0029	·0016	·0007	·0001

The locus of the point D, the point of intersection of W with R_1 and R_2, is a straight line parallel to the line joining AB, and the height q is found from similar triangles—

$$\frac{q - y_1}{c - b} = \frac{V_1}{H}$$

$$\therefore\ q = y_1 + \frac{V_1}{H}(c - b) = \frac{6}{5}r.$$

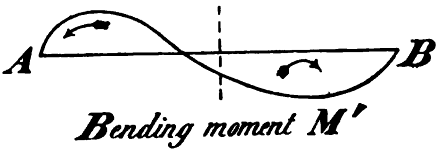


FIG. 435.

The bending moment at the left abutment is—

$$M' = Hy_1.$$

At the right abutment—

$$M'' = Hy_2$$

$$\therefore\ M' = \frac{c}{16}(1 - z^2)(1 + z)(1 - 5z)W.$$

he influence line for M′ is shown in Fig. 435.

TABLE LXXIX.

BENDING MOMENT AT ABUTMENT.

Figures are for c = unity.

$$M' = \frac{c}{16}(1 - z^2)(1 + z)(1 - 5z).$$

z	0	·01	·02	·03	·04	·05	·06	·07	·08	·09
0	+ ·0625	+ ·0599	+ ·0573	+ ·0547	+ ·0519	+ ·0491	+ ·0463	+ ·0433	+ ·0402	+ ·0371
·1	+ ·0340	+ ·0309	+ ·0276	+ ·0243	+ ·0210	+ ·0176	+ ·0142	+ ·0107	+ ·0071	+ ·0036
·2	– ·0000	– ·0036	– ·0073	– ·0109	– ·0146	– ·0183	– ·0220	– ·0258	– ·0295	– ·0333
·3	– ·0370	– ·0407	– ·0444	– ·0481	– ·0518	– ·0555	– ·0592	– ·0629	– ·0666	– ·0702
·4	– ·0735	– ·0770	– ·0806	– ·0838	– ·0875	– ·0903	– ·0935	– ·0965	– ·0995	– ·1025
·5	– ·1055	– ·1083	– ·1110	– ·1136	– ·1160	– ·1182	– ·1204	– ·1226	– ·1245	– ·1263
·6	– ·1280	– ·1295	– ·1308	– ·1320	– ·1332	– ·1340	– ·1347	– ·1351	– ·1353	– ·1355
·7	– ·1355	– ·1351	– ·1346	– ·1339	– ·1331	– ·1316	– ·1301	– ·1284	– ·1263	– ·1239
·8	– ·1214	– ·1188	– ·1154	– ·1120	– ·1084	– ·1042	– ·1000	– ·0951	– ·0900	– ·0848
·9	– ·0790	– ·0728	– ·0663	– ·0595	– ·0522	– ·0445	– ·0365	– ·0288	– ·0191	– ·0098

$$\text{Also } M'' = \frac{c}{16}(1 - z)(1 - z^2)(1 + 5z)W.$$

We may draw the influence line and tabulate values of M″ in terms of z as before.

The bending moment at the crown is—

$$\begin{aligned} Mc &= V_1c - H(r - y_1) - Wb \\ &= \frac{c}{32}(3 - 16z + 18z^2 - 5z^4)W. \end{aligned}$$

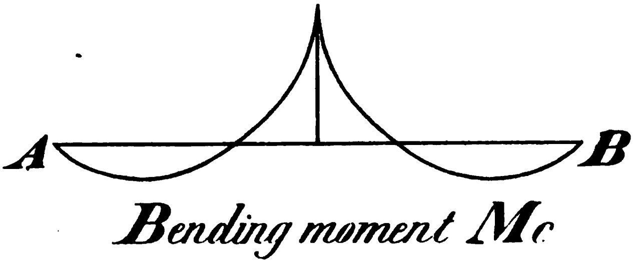


FIG. 436.

The influence line for the bending moment at the crown is shown in Fig. 436.

TABLE LXXX.

BENDING MOMENT AT CROWN Mc.

Figures are for c = unity.

$$Mc = \frac{c}{32}(3 - 16z + 18z^2 - 5z^4).$$

z	0	·01	·02	·03	·04	·05	·06	·07	·08	·09
0	+ ·0938	+ ·0888	+ ·0840	+ ·0793	+ ·0746	+ ·0701	+ ·0658	+ ·0615	+ ·0573	+ ·0533
·1	+ ·0494	+ ·0455	+ ·0418	+ ·0382	+ ·0347	+ ·0313	+ ·0280	+ ·0249	+ ·0218	+ ·0188
·2	+ ·0160	+ ·0133	+ ·0106	+ ·0081	+ ·0056	+ ·0033	+ ·0011	− ·0011	− ·0031	− ·0050
·3	− ·0069	− ·0086	− ·0103	− ·0118	− ·0133	− ·0147	− ·0160	− ·0172	− ·0183	− ·0193
·4	− ·0203	− ·0211	− ·0219	− ·0226	− ·0232	− ·0238	− ·0242	− ·0246	− ·0249	− ·0251
·5	− ·0254	− ·0255	− ·0256	− ·0256	− ·0255	− ·0254	− ·0252	− ·0250	− ·0247	− ·0244
·6	− ·0240	− ·0236	− ·0231	− ·0226	− ·0221	− ·0215	− ·0209	− ·0202	− ·0196	− ·0189
·7	− ·0182	− ·0174	− ·0166	− ·0159	− ·0151	− ·0143	− ·0135	− ·0127	− ·0119	− ·0111
·8	− ·0103	− ·0095	− ·0087	− ·0079	− ·0071	− ·0064	− ·0057	− ·0050	− ·0044	− ·0037
·9	− ·0031	− ·0026	− ·0021	− ·0016	− ·0012	− ·0009	− ·0006	− ·0003	− ·0001	− ·0000

The bending moment at the point E, $\frac{1}{4}l$ from A, is found if r_1 denotes the rise of the arch at the point E $= ar$; then—

$$Me = \frac{V_1c}{2} - H(r - y_1) - W\left(b - \frac{c}{2}\right) \text{ from A to E}$$

$$\text{and } Me = \frac{V_1c}{2} - H(r - y_1). \quad . \quad . \quad \text{,, E ,, B.}$$

These become from A to E—

$$Me = \frac{1}{4}(2 - z)(1+z)^2\frac{c}{2} - \frac{15}{32}(1 - z^2)^2c\left(a - \frac{2}{15}\frac{1-5z}{1-z}\right) - c\left(z - \frac{1}{2}\right).$$

From E to B—

$$Me = \frac{1}{4}(2 - z)(1+z)^2\frac{c}{2} - \frac{15}{32}(1 - z^2)^2c\left(a - \frac{2}{15}\frac{1-5z}{1-z}\right).$$

The influence line for Me is shown in Fig. 437.

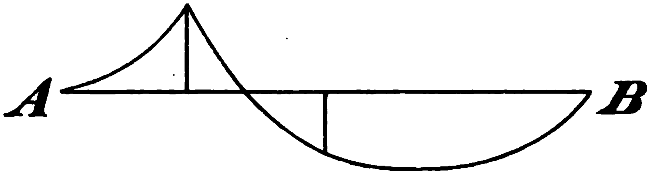


FIG. 437.

Tables LXXVII. to LXXX. were worked out by Mr. Hawken.[1]

The Effect of the Horizontal Thrust in Producing Shortening of the Arched Rib (Fig. 438).—The effect of the horizontal thrust in this case is similar to that for the two-hinged arch, but in consequence of the fixing of the ends the horizontal thrust is applied

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[1] See Appendix I. for Tables worked out by Mr. H. S. Mort, B.Sc., B.E.

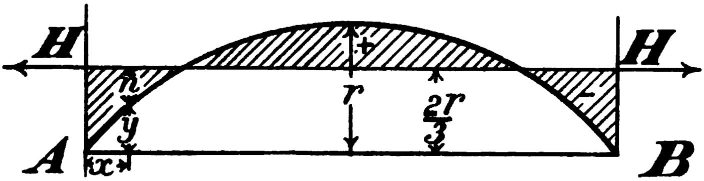

FIG. 488.

above the line AB. Since the loading is symmetrical, the lines of action of H must coincide, and $y_1 = y_2$ also, since—

$$\int_o^l \frac{\mathrm{M}ds}{\mathrm{EI}} = 0, \quad \int_o^l (y - y_1)dx = 0, \text{ or } y_1 = y_2 = \frac{2r}{3}$$

and the moment M_2 is $\mathrm{H}n = -\mathrm{H}\left(y - \frac{2r}{3}\right)$. The horizontal displacement due to H is—

$$\int \frac{\mathrm{M}_2 y ds}{\mathrm{EI}},$$

and the displacement due to the thrust along the axis is—

$$\frac{\sigma l}{\mathrm{E}}$$

$$\therefore \; -\mathrm{H}\int_o^l \frac{\left(y - \frac{2r}{3}\right) ydx}{\mathrm{EI}_c} = \frac{\sigma l}{\mathrm{E}}$$

$$\text{and } \mathrm{H} = -\frac{45\sigma \mathrm{I}_c}{4r^2}$$

which gives the value of H due to rib-shortening. Comparing this value with the corresponding value obtained for the two-hinged arch, we see that it is just six times as great. The maximum ordinate n is $\frac{2r}{3}$ at A and B, and $\frac{r}{3}$ at the crown; from which it follows that the maximum positive moment is twice as great in the arch fixed at the ends as in the arch hinged at the ends due to rib-shortening; also the negative bending moment at A and B due to the same cause is four times as great as that at the crown in the arch hinged at the ends.

The Effect of Temperature.—The lines of action of H due to changes in temperature must coincide, and the distance above AB must be $\frac{2}{3}r$, as in the case for rib-shortening. For a fall of t degrees in temperature the axis would be shortened by an amount etl if it were free to move, hence, making the same assumptions as before, we have—

$$- \mathrm{H}\int_o^l \left(y - \frac{2r}{3}\right) ydx = \mathrm{EI}_c etl$$

$$\text{and } \mathrm{H} = - \frac{45\mathrm{EI}_c et}{4r^2}$$

and, in a similar manner, for a rise of t degrees—

$$\mathrm{H} = + \frac{45\mathrm{EI}_c et}{4r^2}.$$

These values are six times as great as those found for the arch with two hinges. The bending moments due to change of temperature are twice as great at the crown and four times as great at the fixed ends as at the crown of the arched rib hinged at the ends.

General Remarks on Arches.—The calculations in regard to the stresses in arched ribs are determined, in the case of the three-hinged arch, in a manner which is as satisfactory in every respect as that for a simple beam or truss. The three-hinged arch is a statically determinate structure, but temperature stresses occur from the increase or decrease in the horizontal thrust due to a change in temperature, and they modify the existing stresses to a slight extent generally negligible. The effect of temperature produces stresses in the rigid arch from two to four times as great as in the two-hinged arch, but the stiffness is in the reverse direction, being greatest in the rigid arch and least in the three-hinged arch. The theory of the two-hinged and the rigid arch is based upon the deformations which occur within the elastic limit of the material; they are statically indeterminate structures. Prof. Howe, in the case of an arch of 416 ft. span and 67 ft. rise, found that the relative weights of metal in the three types were proportional to the numbers 1·00, 1·21 and 1·30 for the rigid, two-hinged and three-hinged arch respectively.

Erection of Arched Bridges.—In the great viaduct at Müngsten, Germany,[1] having a central arch of 558 ft. without hinges, the structure was erected on the cantilever principle, Fig. 439.

The arch was anchored back by means of the top member EC and anchor bars K. The bar LE was inserted and the holding back ties connected at E. In order to prevent any sagging of the arch, the pier OL, before fixing the ties at E, was turned away from the arch round the point ⊕, and the point O was correspondingly raised by means of suitable supports. Also the member O – 1 could be strained or released as required by means of hydraulic jacks inserted at the point O. In order to close the arch in the manner assumed in the calculations, a temporary hinge was inserted at the point 11 and the arch was then supported at the three points ⊕, 11 and ⊕, Fig. 439, and acted as a three-hinged arch. This was accomplished in detail by inserting hydraulic presses at the crown to reduce the stress in the member O – 1, and with the assistance of those at O to release entirely the stress in O – 1. In order to bring about the final state of the arch without hinges, a steel casting was fitted at the crown, and the members O – 1 inserted after the calculated stresses had been artificially applied by means of the hydraulic presses. In the two-hinged arch of 840 ft. span at Niagara, U.S. America, the structure was made to act temporarily as a three-hinged arch, and afterwards hydraulic presses were inserted at the crown, forcing the ends asunder, and imparting to them the stresses calculated for the two-hinged arch; the gap was then closed by a casting of suitable shape. In the Victoria Falls Bridge,[1] a two-hinged spandrel braced parabolic arch was adopted of 500 ft. span, and 90 ft. rise; the method of imparting the calculated stresses at the crown was similar to that adopted at Niagara. These three examples suffice to prove the possibility of building up statically indeterminate structures in perfect agreement with the assumptions made in the design, and with a sufficient degree of safety.

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[1] *A Hundred Years of German Bridge Building*, by G. C. Mehrtens. Berlin, Julius Springer.

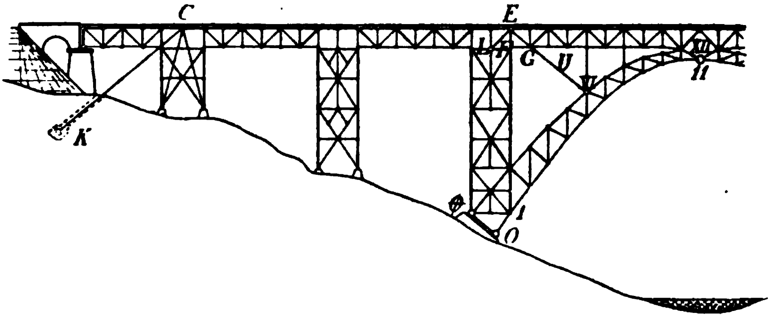

FIG. 439.

Example of the Design of a Three-hinged Arch Bridge.—The following example of a three-hinged parabolic arched rib will be completely worked out, and the bridge partly designed to carry a roadway 18 ft. wide. The bridge, illustrated in Plates IV., IV.*a* and IV.*b*, consists of two parabolic ribs, 147 ft. 2 in. span, and 20 ft. 7¼ in. rise between centres of hinges. The deck is carried by two longitudinal girders supported by vertical columns, which stand upon the top flanges of the arched ribs. Both the longitudinal girders and the arched ribs are spaced 16 ft. apart, centre to centre, and the vertical columns divide the arch into bays of 10 ft., excepting the two central bays, which are each 12 ft. 6 in.

The deck is formed of steel troughing laid transversely on top of the longitudinal girders filled in with coke concrete to a level of 2 in. over the upper surface of the trough. Above the concrete is spread tarred basalt metal to a depth of 5 in. at the centre and 3 in. at the sides. The ends of the troughing are protected by facia plates secured to it by angle irons.

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[1] *Proc. Inst. C.E.*, vol. clxx.

The proportions of the coke concrete are—4 of coke broken to a gauge of 1 in., 2 of sand, and 1 of Portland cement. The weight per cub. ft. of the coke concrete is 90 lbs., whereas stone concrete would weigh 128 lbs., or 30 tons more on the deck of the bridge.

The strength of the troughing may be approximately calculated as follows:—

Dead load, including concrete, metal and troughing 3 ft. wide between longitudinals = 2024 lbs. per foot run.

$$\text{Bending moment} = \frac{Wl}{8} = \frac{2024 \times 3 \times 16}{2240 \times 8} = 5{\cdot}42 \text{ foot tons.}$$

Live load consisting of traction engine shown in Fig. 390.

Bending moment = $4{\cdot}75(8 - 2{\cdot}5) = 26{\cdot}13$ foot tons.

Total bending moment on one trough = 31·55 foot tons = 378 inch tons.

Treating each trough as a girder, with each flange consisting of the horizontal portion plus one-half the sloping portion above or below the neutral axis, we have—

Area of one flange = $(28{\cdot}5 - 4 \times \frac{3}{4})\frac{5}{16} = \frac{5}{16} \times 25{\cdot}5$ sq. in.

Distance between centres of gravity = 11·375 in.

$$\left.\begin{array}{l}\text{Moment of}\\ \text{resistance}\end{array}\right\} = \frac{25{\cdot}5 \times 5 \times 11{\cdot}375 \times 28}{16} = 2538 \text{ inch tons.}$$

$$\text{Hence the factor of safety is } \frac{2538}{378} = 6{\cdot}7.$$

Longitudinal Girders consist of two plate web girders spaced 16 ft. centres across the bridge, each 18 in. deep, tapering to 6⅜ in. near centre of bridge. They are stiffened over columns with diaphragms, and the bottom flanges are laced with 3 × ⅜ bars. The ends of longitudinals are continued over abutments, and have gun-metal bearing-plates secured on their lower flanges sliding on cast-iron bed-plates secured to bed-stones.

Each longitudinal girder is calculated as a continuous girder of five equal spans, the two central bays being supported throughout their length by the rib.

The loads on the longitudinal girder are as follows:—

Live load, taken as 84 lbs. per sq. ft. of roadway = 0·35 tons per lineal ft. on each girder.

Dead load = 0·7 tons per lineal ft. on each girder.

The maximum bending moments for these loads occur when the bays x_1, x_3 and x_5 are fully loaded, and bays x_2 and x_4 are loaded with the dead load only.

The bending moments at centres of bays x_1, x_3 and x_5, Plate IV., may be found, by the methods already explained, to be 13 foot tons, and at centres of bays x_2 and x_4 to be 8·7 foot tons.

The traction engine produces the greatest bending moment when the load on one wheel is carried entirely by one girder and $\frac{11 \cdot 33}{16}$ of the other also.

The total load on one girder due to the driving wheels = $4 \cdot 75 \left(1 + \frac{11 \cdot 33}{16}\right) = 8 \cdot 11$ tons, and the load due to the trailing wheels is $3 \cdot 25 \left(1 + \frac{11 \cdot 33}{16}\right) = 5 \cdot 55$ tons.

Bending moment at centre of bay due to driving wheels = 20·3 foot tons.

Bending moment at centre of bay due to trailing wheels = 13·0 foot tons.

Bending moment at centre of bay due to dead load = 8·75 foot tons.

The results are summarised as follows, the working stresses being taken as 4 and 5 tons compression and tension respectively:—

TABLE LXXXI.

Flange.	Effective area.			
	Driving wheels over bay x_1 trailing over abutment.		Driving wheels over centre of bay x_2 trailing over x_1.	
	Required.	Provided.	Required.	Provided.
	sq. in.	sq. in.	sq. in.	sq. in.
Top	$\left\{\frac{20}{1 \cdot 42 \times 4}\right. = 3 \cdot 5$	$\left\{\begin{matrix}\text{Two angles} \\ 4 \times 3 \times \frac{1}{2}\end{matrix}\right\} = 6 \cdot 5$	$\left\{\frac{21}{1 \cdot 42 \times 5}\right. = 3 \cdot 0$	$\left\{\begin{matrix}\text{Two angles} \\ \text{as before} \\ \text{less rivets}\end{matrix}\right. = 5 \cdot 75$
Bottom	$\left\{\frac{20}{1 \cdot 42 \times 5}\right. = 2 \cdot 8$	$\left\{\begin{matrix}\text{Do. less} \\ \text{rivets}\end{matrix}\right\} = 5 \cdot 75$	$\left\{\frac{21}{1 \cdot 42 \times 4}\right. = 3 \cdot 7$	Do. gross = 6·5

Maximum shear due to dead and live load = 11·61 tons.

The web plate is made $\frac{3}{8}$ in. thick, and the rivets $\frac{3}{4}$ in. diameter, 4 in. pitch, which may easily be shown to be safe.

The columns are calculated by the formula—

$$10{,}000 - 45\frac{l}{r} \text{ for live load,}$$

$$20{,}000 - 90\frac{l}{r} \text{ for dead load,}$$

r being the least radius of gyration of the cross-section. They are made larger than necessary for appearance.

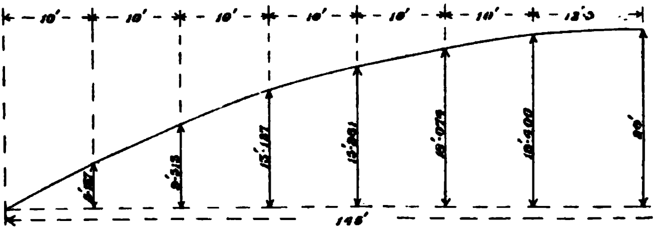


Fig. 440.

Column 1 requires by above formula 4·6 sq. in. Area provided is 13·2 sq. in.
„ 2 „ „ 5·3 „ „ „ 13·2 „
„ 3 „ „ 4·1 „ „ „ 10·2 „
„ 4 „ „ 3·6 „ „ „ 10·2 „

Arched Ribs.—The load resting upon the arched ribs may now be ascertained.

TABLE LXXXII.

Point number.	Dead load.			Distributed live load equals dead load $x\frac{0{\cdot}35}{0{\cdot}7} = \frac{1}{2}$
	Deck and longitudinal girder.	Column and bracing of rib.	Total.	
	tons.			
1	$0{\cdot}7 \times 10 \times \frac{15}{38} = 2{\cdot}8$	2·1	4·9	1·4
2	$\times \frac{43}{38} = 8{\cdot}0$	2·8	10·8	4·0
3	$\times \frac{37}{38} = 6{\cdot}8$	2·3	9·1	3·4
4	$\times \frac{37}{38} = 6{\cdot}8$	2·0	8·8	3·4
5	$\times \frac{43}{38} = 8{\cdot}0$	1·8	9·8	4·0
6	$\times \frac{15}{38} + 0{\cdot}7 \times 0{\cdot}5 = 6{\cdot}3$	1·6	7·9	3·2
7	$0{\cdot}7 \times 11{\cdot}25 = 7{\cdot}9$	1·7	9·6	3·9
8	$0{\cdot}7 \times 6{\cdot}25 = 4{\cdot}4$	1·2	5·6	2·2
	Totals . .		66·5	25·5

The bending moments in the arched rib considered as 145 ft. span, with a rise of 20 ft. (Fig. 440) for live and dead load, may now be calculated, using the formula $b = \dfrac{al}{2l + a}$ to find the distribution of loads which will give the maximum; thus for point 2—

$$b = \frac{62{\cdot}5 \times 72{\cdot}5}{2 \times 72{\cdot}5 + 62{\cdot}5} = 21{\cdot}8 \text{ ft.}$$

For Maxima Bending Moment. Positive (see Fig. 441)—

$$0 = 72{\cdot}5V - 20H + 10{\cdot}8 \times 10 + 9{\cdot}1 \times 20 + 8{\cdot}8 \times 30 + 9{\cdot}8 \times 40 + 7{\cdot}9 \times 50 + 13{\cdot}5 \times 60 + 7{\cdot}8 \times 72{\cdot}5.$$

$$0 = 72{\cdot}5V + 20H - 14{\cdot}8 \times 10 - 12{\cdot}5 \times 20 - 12{\cdot}2 \times 30 - 13{\cdot}8 \times 40 - 11{\cdot}1 \times 50 - 13{\cdot}5 \times 60 - 7{\cdot}8 \times 72{\cdot}5$$

$\therefore H = 149{\cdot}1$, and $V = 3{\cdot}65$.

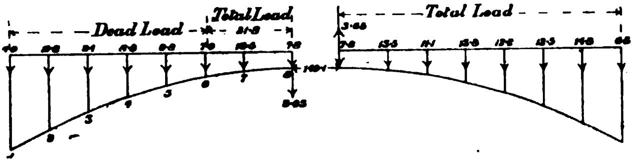


FIG. 441.

Using these values, and taking moments about the neutral axis of the rib at 2, we have—

$$M_2 = 3{\cdot}65 \times 62{\cdot}5 - 149{\cdot}1 \times 14{\cdot}86 + 7{\cdot}8 \times 62{\cdot}5 + 13{\cdot}5 \times 50 + 7{\cdot}9 \times 40 + 9{\cdot}8 \times 30 + 8{\cdot}8 \times 20 + 9{\cdot}1 \times 10 = +\ 52{\cdot}01.$$

Negative (see Fig. 442)—

$$0 = -\ 72{\cdot}5V - 20H + 14{\cdot}8 \times 10 + 12{\cdot}5 \times 20 + 12{\cdot}2 \times 30 + 13{\cdot}8 \times 40 + 11{\cdot}1 \times 50 + 9{\cdot}6 \times 60 + 5{\cdot}6 \times 72{\cdot}5$$

$$0 = -\ 72{\cdot}5V + 20H - 10{\cdot}8 \times 10 - 9{\cdot}1 \times 20 - 8{\cdot}8 \times 30 - 9{\cdot}8 \times 40 - 7{\cdot}9 \times 50 - 9{\cdot}6 \times 60 - 5{\cdot}6 \times 72{\cdot}5$$

$\therefore H = 129{\cdot}4$, and $V = 3{\cdot}65$

$$M_2 = -\ 3{\cdot}65 \times 62{\cdot}5 - 129{\cdot}4 \times 14{\cdot}86 + 5{\cdot}6 \times 62{\cdot}5 + 9{\cdot}6 \times 50 + 11{\cdot}1 \times 40 + 13{\cdot}8 \times 30 + 12{\cdot}2 \times 20 + 12{\cdot}5 \times 10$$

$$= -\ 94$$

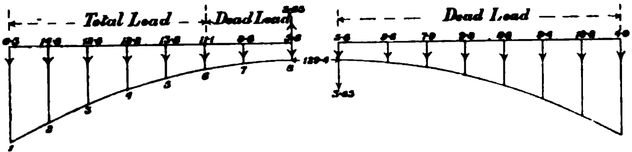


FIG. 442.

In the same manner it may be shown that for the point 3 the load boundary = 19·3 ft. For maximum positive bending moments $H = 149{\cdot}1$ and $V = 3{\cdot}65$; $M_3 = +\ 98{\cdot}07$. For maximum negative bending moments $H = 129{\cdot}4$, $V = 3{\cdot}65$, and $M_3 = -\ 140{\cdot}03$.

For the point 4 the load boundary is 16·3 ft., and we obtain the same values of H and V as for points 2 and 3. $M_4 = +123{\cdot}31$ and $-158{\cdot}10$.

For point 5 the load boundary is 12·3 ft.; $H = 143{\cdot}23$, $V = 5{\cdot}21$, and $M_5 = +118{\cdot}04$; also $H = 135{\cdot}25$, $V = 5{\cdot}21$, and $M_5 = -153{\cdot}03$.

For point 6 the load boundary is 8·8 ft., the values of H and V are the same as for point 5; and $M_6 = +112{\cdot}29$ and $-117{\cdot}25$.

For point 7 the load boundary is 4·7 ft., the values of H and V are the same as for points 5 and 6; and $M_7 = +78{\cdot}12$ and $-74{\cdot}92$.

Stresses due to Dead Load and Traction Engine.—The maximum loads carried by one rib due to a traction engine rolling over the bridge will be the same as in the longitudinal girder, *viz.*, 8·11 tons concentrated on the driving, and 5·55 tons on the trailing, wheels.

Let the driving wheels be over column 2, Fig. 443.

Then—

$$
\begin{aligned}
0 &= -72{\cdot}5V - 20H + 18{\cdot}91 \times 10 + 9{\cdot}1 \times 20 + 8{\cdot}8 \times 30 + 9{\cdot}8 \\
&\qquad \times 40 + 7{\cdot}9 \times 50 + 9{\cdot}6 \times 60 + 5{\cdot}6 \times 72{\cdot}5 \\
0 &= -72{\cdot}5V + 20H - 10{\cdot}8 \times 10 - 9{\cdot}1 \times 20 - 8{\cdot}8 \times 30 - 9{\cdot}8 \\
&\qquad \times 40 - 7{\cdot}9 \times 50 - 9{\cdot}6 \times 60 - 5{\cdot}6 \times 72{\cdot}5 \\
\therefore H &= 118{\cdot}18 \text{ and } V = 0{\cdot}56 \\
M_2 &= -0{\cdot}56 \times 62{\cdot}5 - 118{\cdot}18 \times 14{\cdot}86 + 5{\cdot}6 \times 62{\cdot}5 + 9{\cdot}6 \times 50 \\
&\qquad + 7{\cdot}9 \times 40 + 9{\cdot}8 \times 30 + 8{\cdot}8 \times 20 + 9{\cdot}1 \times 10 \\
&= -84{\cdot}15
\end{aligned}
$$

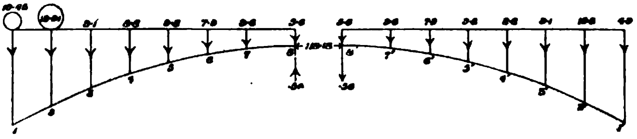


FIG. 443.

The maximum negative bending moment at point 2 is therefore less for the traction engine driving wheel over the point 2 than for the dead and distributed live loads when the latter occupies the worst position, but it will be found that the maxima negative moments occur at the various points in nearly every other case when the driving wheel of the traction engine is over these points respectively. So far as this example is concerned, the negative bending moments for the dead and distributed live loads need not have been considered, since the traction engine always produces the greater effect.

The results are given in Table LXXXIII.

The bending moments recorded in the following table, which have been calculated with reference to the neutral axis of the rib, must be divided by the effective depth of the rib in order to find the stresses in the flanges due to bending. The positive sign denotes the upward bending moments, and the negative sign the downward bending moments, so that the former produces tension in the extrados and compression in the intrados, while the latter produces compression in the extrados and tension in the intrados.

The horizontal thrust and the vertical reactions and loads produce compressive stresses in the rib which are independent of the depth, and these must be combined with the stresses due to bending in order to find the resultant stresses.

Let F = the sum of the vertical forces on the right side of the neutral axis of the rib, including the vertical reaction.

a = the angle which the tangent to the neutral axis makes with the horizon.

H = the horizontal thrust due to the vertical forces.

Then the compressive stress in the extrados and intrados of the rib is—

$$\frac{\text{F} \sin a + \text{H} \cos a}{2}$$

The shearing stress is—

$$\text{F} \cos a - \text{H} \sin a.$$

If we calculate the values of sin a and cos a at the various points in the neutral axis corresponding with the points about which the bending moments were taken, we can find the values of the compressive and shearing stresses and combine these with the stresses due to bending.

Remembering that the neutral axis of the rib is parabolic, and that the subtangent is twice the horizontal distance from the point at which the tangent is drawn to the centre of the hinge at the springing, we can tabulate the values of the sine and cosine as in the following table. The effective depth of the web is 2·5 ft.

The maximum stress due to bending is 72·18 tons, therefore the sectional area of the top and bottom flanges is—

$$\frac{72 \cdot 18}{4} = 18 \cdot 045 \text{ sq. in.}$$

The area provided (Plate IV.) is—

Four angles 4 × 3 × ½	13·00 sq. in.
Two plates 9 × ⅜	6·75 „
Total area of one flange	19·75 „
Total area of both flanges	39·50 „

The maximum resultant is 138 tons compression, which is distributed over about 28 in. of the depth of the rib, as a stress uniformly

TABLE LXXXIII.

Number of point.	Maxima moments due to the dead and partially distributed live load.		Maxima moments due to the dead load and the traction engine.														Maxima moments due to any manner of loading.	
			Rolling to the right.							Rolling to the left.								
			Driving wheels over							Driving wheels over								
			2	3	4	5	6	7	8	2_1	3_1	4_1	5_1	6_1	7_1	8_1		
2	+ 52·01	- 94·00	- 84·15	- 112·63	- 85·30	- 58·89	- 31·86	- 4·69	+ 28·33	+ 14·15	+ 6·23	- 2·00	- 10·56	- 18·48	- 26·56	- 36·83	+ 52·01	- 112·63
3	+ 98·07	- 140·03	- 70·10	- 155·33	- 159·25	- 108·40	- 77·03	- 5·55	+ 57·65	+ 11·81	- 2·17	- 15·85	- 29·85	- 43·33	- 56·91	- 73·95	+ 98·07	- 159·25
4	+ 123·31	- 158·10	- 53·70	- 117·14	- 180·45	- 163·26	- 90·07	- 16·84	+ 73·37	+ 6·10	- 10·36	- 26·95	- 43·83	- 60·29	- 76·81	- 97·47	+ 123·31	- 180·45
5	+ 118·04	- 153·03	- 40·28	- 84·58	- 128·80	- 173·42	- 136·60	- 44·28	+ 69·69	+ 3·88	- 12·92	- 29·80	- 46·93	- 63·73	- 80·57	- 101·59	+ 118·04	- 173·42
6	+ 112·29	- 117·25	- 18·69	- 46·44	- 74·15	- 102·12	- 129·88	- 76·51	+ 58·00	- 5·51	- 21·06	- 35·65	- 50·54	- 64·98	- 79·55	- 97·70	+ 112·29	- 129·88
7	+ 78·12	- 74·92	- 6·73	- 20·49	- 34·25	- 48·14	- 51·91	- 75·67	+ 20·71	- 7·3	- 17·01	- 26·75	- 36·61	- 46·34	- 56·08	- 68·21	+ 78·12	- 75·67

TABLE LXXXIV.

Point number.	Bending moments +	Stresses due to bending moments.		Sin a.	Cos a.	Sum of vertical forces. F.	Horizontal thrust. H.	$\frac{F \sin a + H \cos a}{2}$		Resultant stresses.		Shearing stresses F cos a − H sin a.
		Extrados tension.	Intrados compression.					Extrados compression.	Intrados compression.	Extrados compression.	Intrados tension.	
2	52·01	20·80	20·80	0·249	0·969	71·35	149·10	81·12	81·12	60·32	101·92	32·01
3	98·07	39·23	39·23	0·231	0·973	60·55	149·10	79·53	79·53	40·30	118·76	23·08
4	123·31	49·32	49·32	0·214	0·977	51·45	149·10	78·34	78·34	29·02	127·66	18·36
5	118·04	47·22	47·22	0·196	0·981	40·31	143·23	74·21	74·21	26·99	121·43	11·47
6	112·29	44·92	44·92	0·178	0·984	30·51	143·23	73·18	73·18	28·26	118·10	4·53
7	78·12	31·25	31·25	0·160	0·987	22·61	143·23	72·49	72·49	41·24	103·74	− 0·60

TABLE LXXXV.

Point number.	Bending moments −	Stresses due to bending moments.		Sin a.	Cos a.	Sum of vertical forces. F.	Horizontal thrust. H.	$\frac{F \sin a + H \cos a}{2}$		Resultant stresses.		Shearing stresses F cos a − H sin a.
		Extrados compression.	Intrados tension.					Extrados compression.	Intrados compression.	Extrados compression.	Intrados compression and tension.	
2	112·63	45·05	45·05	0·249	0·969	73·68	121·54	68·06	68·06	113·11	+ 23·01	41·13
3	159·25	63·70	63·70	0·231	0·973	62·08	125·01	67·99	67·99	131·69	+ 4·29	31·53
4	180·45	72·18	72·18	0·214	0·977	47·43	125·01	66·14	66·14	138·32	− 6·04	19·59
5	173·42	69·37	69·37	0·196	0·981	37·63	128·41	66·67	66·67	136·04	− 2·70	11·75
6	129·88	51·95	51·95	0·178	0·984	26·88	131·84	67·26	67·26	119·21	+ 15·31	2·98
7	75·67	30·27	30·27	0·160	0·987	18·04	135·25	68·19	68·19	98·46	+ 37·92	− 3·83

varying from 0 to a maximum. If we take half the depth of th as effective, and add the area to the area of one flange, we ha the two web plates, taken as $\frac{1}{2}$ in. thick, 15 sq. in., or a total a 34·5 sq. in.; hence the intensity of stress is about—

$$\frac{138}{34\cdot 5} = 4 \text{ tons per sq. in.}$$

The maximum shearing stress is 41·13 tons, or $\frac{41\cdot 13}{2\cdot 5} = 16\cdot 4$ t per ft. on the two web plates, or 8·2 tons on each web plate.

Assume that the rivets are $\frac{7}{8}$ in. diameter and 4 in. pitch, then t intensity of shearing stress on the rivets is—

$$3 \times 0\cdot 6 \times 2 \times f = 8\cdot 2$$
$$\therefore f = 2\cdot 3 \text{ tons per sq. in.}$$

The intensity of pressure on the bearing area is—

$$p \times 3 \times \tfrac{1}{2} \times \tfrac{7}{8} = 8\cdot 2$$
$$\therefore p = 6\cdot 2 \text{ tons per sq. in.}$$

The intensity of shearing stress on the web is—

$$\frac{8\cdot 2}{(12 - 2 \times \frac{7}{8})\frac{1}{2}} = 1\cdot 6 \text{ tons per sq. in.}$$

Hence there is no danger of buckling, and the sizes of rivets, pitch and thickness of web may be adopted.

The section of the web is made uniform throughout, and for convenience of carriage the joints occur in the centre of each bay.

Hinges.—The hinges at the centre and springing are formed each of two castings working on a steel pin 7 in. in diameter and 2 ft. $4\frac{1}{4}$ in. long. The maximum pressure on the hinges occurs when the bridge is fully loaded. In this case H = 162·3 tons and V = 0.

At the centre the pressure is 162·3 tons, and the intensity of pressure is—

$$\frac{162\cdot 3}{7 \times 28\cdot 25} = 0\cdot 82 \text{ tons per sq. in.}$$

At the springing—

$$F \sin a + H \cos a = 92 \times 0\cdot 481 + 162\cdot 3 \times 0\cdot 875 = 186\cdot 26 \text{ tons.}$$

The intensity of pressure per sq. in. is—

$$\frac{186\cdot 26}{7 \times 28\cdot 25} = 0\cdot 94 \text{ tons.}$$

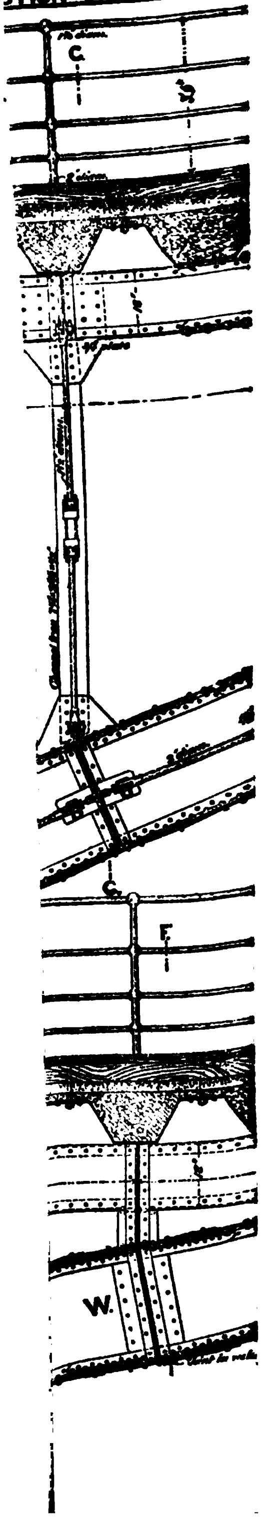
The intensity of compressive stress on the rib at the springing when the bridge is fully loaded is—

$$\frac{186\cdot 26}{69\cdot 5} = 2\cdot 7 \text{ tons nearly.}$$

The details of the bridge are sufficiently illustrated in Plates IV., IV.*a* and IV.*b*.

ARCH

CTION ON C LINE.



IN

TA

The calculations for wind pressure present no difficulty; the sections adopted for the wind braces are shown on the Plate IV.*a*.

Abutments.—The abutments are composed of sandstone ashlar masonry, set in cement mortar, with wing walls and pilasters; inequalities in the rock face may be filled with cement concrete. The bedstones or skewbacks should be made of granite or other suitable stone.

Erection of the Ribs.—The ribs may be riveted on a staging, so that the longitudinal girders may have, when fixed in position, an even fall of 3 in. from the central hinge to the end girder when the temperature in the sun is 70° Fahr. Should the ribs be erected on a warmer or colder day, the necessary allowance for temperature must be made in wedging up the end hinges.

The foregoing method of calculation may be used for either a plate web or lattice web; if the web is designed with a single system of triangulation as in Fig. 408, the stresses may be obtained graphically as shown in Fig. 409, or by passing a section through any three members of the rib and taking moments in the usual way.

Suspension Bridges consisting of Steel Wire Cables stiffened near the Towers by means of Inclined Stays.—This is not a satisfactory construction as ordinarily carried out, as at best the stays leave the centre portion of the span unsupported, and the lengths of the inclined stays cannot be adjusted so that they will act efficiently under variations of temperature; the only useful purpose they serve is to check the vibrations and oscillations produced by the rolling load.

Suspension Bridges consisting of Steel Cables with a Stiffening Girder.—The steel cables may be combined with a horizontal stiffening girder, in which case the cables must sustain the whole live and dead load, the girder merely acting when the bridge is partially loaded, and the effect is to distribute the load over the cables.

The cables tend to adapt themselves to the form of the equilibrium curve for the load they sustain. If the load is equally distributed along the horizontal, the curve of equilibrium may be proved to be a parabola.

The suspension rib is the inverted form of the arch rib, in which the tension in the suspended rib, or cables, corresponds with the compression in the arch rib.

Let ACB, Fig. 444, represent a cable suspended from the points A and B at the top of two towers and loaded uniformly over the horizontal distance between A and B with w per unit of length.

Take moments about B, then—

$$Hr - \frac{wl}{2}\left(\frac{l}{4}\right) = 0, \text{ or } H = \frac{wl^2}{8r},$$

where H denotes the horizontal pull in the cables at C.

Let T denote the tension at B, then—

$$T = \sqrt{H^2 + \left(\frac{wl}{2}\right)^2} = \frac{wl^2}{8r}\sqrt{1 + 16\frac{r^2}{l^2}}.$$

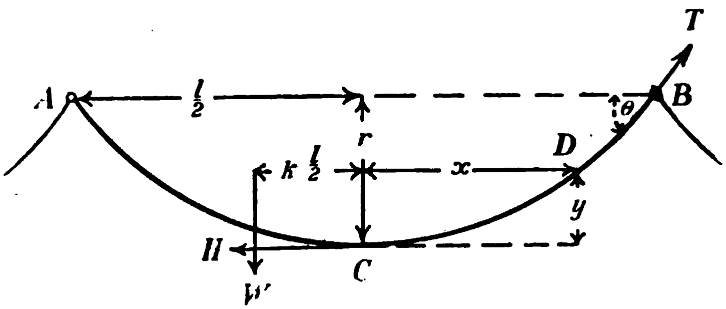


Fig. 444.

Take moments about the point D where the tension is denoted by T_x, then—

$$Hy = \frac{wx^2}{2} \therefore y = \frac{wx^2}{2H} = \frac{4r}{l^2}x^2.$$

This shows that the curve ACB is a parabola—

$$T_x = \sqrt{H^2 + (wx)^2} = H\sqrt{1 + 64\frac{r^2x^2}{l^4}}.$$

The length of the loaded cable is—

$$c = 2\int_0^{\frac{l}{2}}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}dx.$$

$$\text{Where } \frac{dy}{dx} = \frac{wx}{H} = \frac{8rx}{l^2} \therefore c = 2\int_0^{\frac{l}{2}}\left\{1 + \left(\frac{8rx}{l^2}\right)^2\right\}^{\frac{1}{2}}dx$$

$$\therefore c = l\left(1 + \frac{8}{3}\frac{r^2}{l^2} - \frac{32}{5}\frac{r^4}{l^4} + \ldots\right).$$

This gives the length of the loaded cable for the span l and the dip r. If c^1 denote the length of the unloaded cable, the change of length is, $c - c^1 = \delta c$.

Let A denote the area of the wires in the cross-section of the cables, then it can be shown that—

$$\delta c = \frac{Hl}{AE}\left(1 + \frac{16r^2}{3l^2}\right).$$

The length of the unloaded cable is therefore—

$$c^1 - c = \delta c.$$

Let r^1 denote the altered dip of the cable, then—

$$r^1 - r = \delta r,$$

where r denotes the increase in the value of r due to the load wl. If H denotes the horizontal pull due to a rise of temperature, t° and e the co-efficient of expansion, it can be shown that—

$$\delta c = etc\,;\ \delta r = \frac{3}{16}\,\frac{etcl}{r},\ \text{and}\ \delta \mathrm{H} = -\mathrm{H}\frac{\delta r}{r},\ \text{also}\ \delta \mathrm{T} = -\left(1 - \frac{16r^2}{l^2}\right)\mathrm{T}\frac{\delta r}{r}.$$

If a single load is applied at a distance of $\frac{kl}{2}$ from C, Fig. 444, the vertical reactions are—

$$\frac{\mathrm{W}}{2}\left(1 + k\right)\ \text{at A; and}\ \frac{\mathrm{W}}{2}\left(1 - k\right)\ \text{at B.}$$

The horizontal tension at A and B is found by taking moments about C—

$$-\mathrm{H}r + \frac{\mathrm{W}}{2}\left(1 + k\right)\frac{l}{2} - \frac{\mathrm{W}kl}{2} = 0.$$

$$\therefore \mathrm{H} = \frac{\mathrm{W}l}{4r}(1 - k).$$

Let θ denote the angle made by the tangent at D with the horizontal, then—

$$\mathrm{T}_x = \mathrm{H}\sec\theta = \mathrm{H}(1 + \tan^2\theta)^{\frac{1}{2}} = \mathrm{H}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}},$$

but since $y = \frac{4rx^2}{l^2}$ and $\frac{dy}{dx} = \frac{8rx}{l^2}$,

$$\therefore \mathrm{T}_x = \mathrm{H}\left(1 + 64\frac{r^2x^2}{l^4}\right)^{\frac{1}{2}}.$$

The pull at A and B due to W is found by putting $x = \frac{l}{2}$.

Stiffening Truss.—The stiffening truss or girder is assumed to act only when the cables are partially loaded, and the curve of equilibrium is no longer a parabola. There is no stress in the truss for a uniformly distributed load. Stiffening trusses may be continuous between the towers, or they may be hinged at the centre.

Continuous Truss.—Let Fig. 445 represent a stiffening truss without a hinge at the centre C. Let a live load of w per unit of length extend from A to D, and let w' denote the upward pull on the truss per unit of length. R_1 and R_2 the reactions at A and B respectively. Then assuming that the cable coincides with a parabola both for uniform and partial loads, and also that the stresses in the hangers are equal, then—

The upward pull $= w'l = wkl$; or $w' = wk$.

$\mathrm{R}_1 + \mathrm{R}_2 = 0$. Take moments about A, then—

$$-\mathrm{R}_2 l - w'l\,\frac{l}{2} + wkl \times \frac{kl}{2} = 0,$$

$$\therefore \mathrm{R}_1 = \frac{wlk}{2}(1 - k)\,;\ \mathrm{R}_2 = -\frac{wkl}{2}(1 - k).$$

When $k = 0$, or $k = 1$, the reactions are zero; when $k = \frac{1}{2}$, the reactions are each $= \frac{wl}{8}$, which is a maximum.

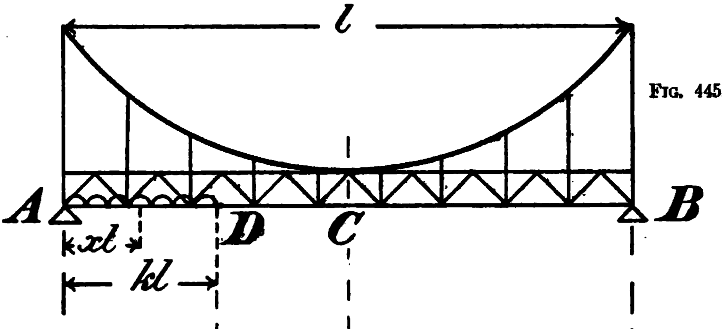


FIG. 445.

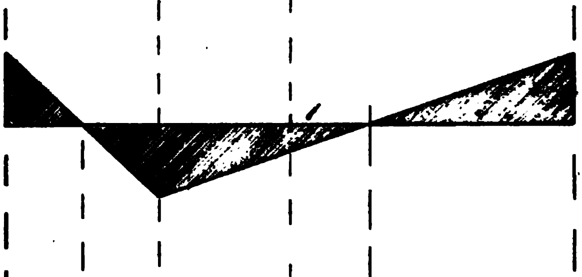


FIG. 446.

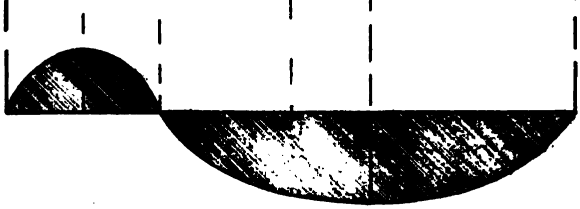


FIG. 447.

The maximum shear at A or B occurs when half the span is loaded, and the shear at any point distant xl from A, Fig. 445, is—

$$S = R_1 + w'xl - wxl = \tfrac{1}{2}wl(k - k^2 + 2kx - 2x).$$

$$S = 0 \text{ when } x = \frac{k}{2}\text{; and } - R_1 \text{ when } x = k.$$

Hence the distribution of shearing stress is represented in Fig. 446. The bending moment at any section xl from A is—

$$M = R_1 xl + w'xl \times \frac{xl}{2} - wxl \times \frac{xl}{2} = \tfrac{1}{2}wl^2(kx - k^2x + kx^2 - x^2).$$

$$\text{When } x = 0,\ M = 0, \text{ when } x = k,\ M = 0.$$

$$\text{When } x = \frac{k}{2}; M = \frac{wl^2}{8}(k^2 - k^3).$$

This is a maximum when $k = \frac{2}{3}$ and $= \frac{1}{54}wl^2$, or for the live load

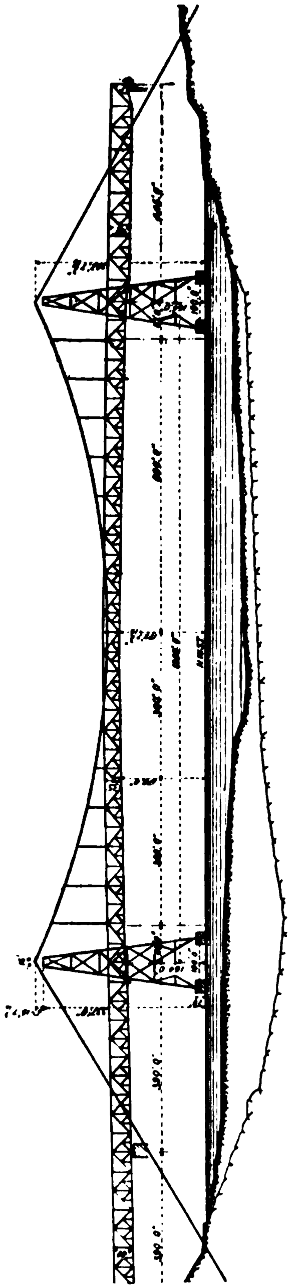


FIG. 448.

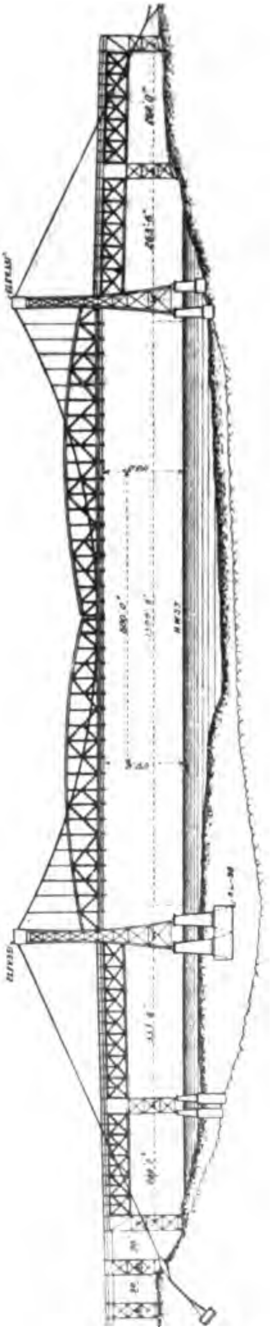


FIG. 452.

covering two-thirds of the span. In a similar manner it may be shown that the greatest negative moment occurs when the live load covers $\frac{1}{3}$ of the span, and its value is $-\frac{1}{54}wl^2$, Fig. 447.

Hence the maximum shear is $\frac{wl}{8}$, and the maximum moment $\frac{wl^2}{54}$. In designing the truss, the top and bottom members are generally made uniform in sectional area; the web is also made uniform in section throughout its length. The assumptions made are sufficiently exact for a heavy cable and a light live load, but they would be incorrect for a comparatively light cable and a heavy live load.

Fig. 448 shows a suspension bridge of 1360 ft. span with a continuous stiffening truss proposed for Sydney Harbour.[1]

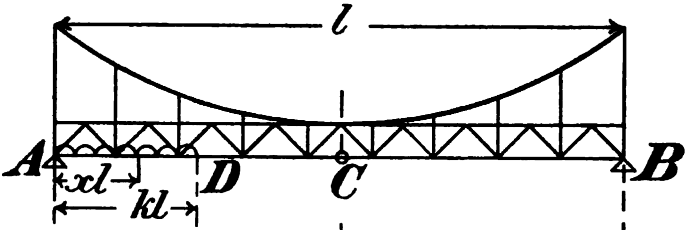


FIG. 449.

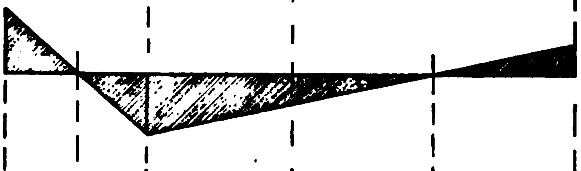


FIG. 450.

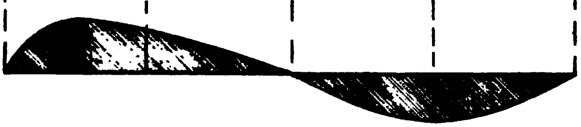


FIG. 451.

Truss Hinged at the Centre.—Fig. 449 represents a stiffening truss hinged at the centre and fixed at A and B. The same assumptions are made as in the truss without a hinge. There is no bending at A, C or B. As before, we take moments about A and C, then—

$$R_2 l + wk\left(\frac{kl}{2}\right)l = 0;\ \text{and}\ \frac{R_1 l}{2} - \frac{R_2 l}{2} - wkl\left(\frac{l}{2} - \frac{kl}{2}\right) = 0;$$

$$\text{also } R_1 + R_2 = wkl - w'l.$$

$$\therefore R_1 = \frac{wl}{2}(2k - 3k^2);\ R_2 = -\frac{wl}{2}k^2;\ w' = 2wk^2.$$

---

[1] Report of the Advisory Board for Sydney Harbour Bridge.

$$R_1 \text{ is a maximum when } \frac{dR_1}{dk} = \frac{wl}{2}(2 - 6k) = 0,$$

$$\text{or } k = \tfrac{1}{3} \therefore R_1 = \frac{wl}{6}.$$

$$R_2 \text{ is a maximum when } k = \tfrac{1}{2} \therefore R_2 = -\frac{wl}{8}.$$

The shear at D is—

$$S = R_1 + w'kl - wkl = \frac{wl}{2}(4k^3 - 3k^2).$$

When $k = \frac{1}{2}$ this is a maximum, and—

$$S = -\frac{wl}{8}.$$

The distribution of shearing stress is shown in Fig. 450. The maximum moments occur at the points where the shear is zero. The shear and bending moment at any point distant xl from A are—

$$S = R_1 - wxl + w'xl;\ M = R_1xl - \frac{wx^2l^2}{2} + \frac{w'x^2l^2}{2}$$

$$\text{when } S = 0,\ xl = \frac{l}{2}\left(\frac{2k - 3k^2}{1 - 2k^2}\right), \text{ and } M = \frac{wl^2}{8}\frac{(2k - 3k^2)^2}{1 - 2k^2}.$$

This value of M will give the maximum moment for any value of k, and the greatest value of this moment occurs when $\frac{dM}{dk} = 0$, from which we obtain—

$$3k^3 - 3k + 1 = 0.$$

One of the values of k is 0·395.

$$\therefore M = 0{\cdot}1506\frac{wl^2}{8} = \frac{wl^2}{53} \text{ about.}$$

The shear and bending moment at any point distant yl from B are—

$$S = R_2 + w'yl;\ M = R_2yl + \frac{w'y^2l^2}{2}$$

$$S = -\frac{wlk^2}{2} + 2wk^2yl, \text{ and } S = 0, \text{ when } y = \frac{l}{4}.$$

For this value of y, $M = -\frac{wl^2}{64}$, which is the maximum negative bending moment due to the live load. The distribution of moments is shown in Fig. 451.

The results obtained are fairly correct for a relatively small live load, but uniform tension in the hangers under partial loads is only approximately true. It is usual to design the web members for a shear of $\frac{wl}{6}$, and the chords for a moment of $\frac{wl^2}{53}$. Suspension bridges having stiffening girders without hinges are generally made much shallower than when hinged at the centre, and in the bridge proposed by G. S. Morrison[1] for a span of 3000 ft. the stiffening girders were made without hinges 66 ft. 8 in. deep. In a bridge of the same span proposed by the Union Bridge Company, New York,[2] the stiffening trusses were hinged at the centre, and 200 ft. deep in the centre of each half-span, the top chords being curved.

Fig. 452 shows a bridge of this type with the stiffening girder hinged at the centre like Fig. 449. It was proposed for Sydney Harbour, and the two designs may be compared.

The following example of a stiffened suspension bridge[3] will illustrate the method of determining the most important of the stresses in this class of structure.

The bridge illustrated in Plates V., V.*a* and V.*b*, is 775 ft. long, in three spans of 150, 500 and 125 ft. respectively, and 28 ft. wide in the clear. It consists of steel-wire cables suspended from masonry towers and supporting steel lattice stiffening girders and steel cross-girders.

The cables, which are six in number, three on each side of the bridge, consist each of seven ropes formed with plough steel wire. The central deflection of the cables is 38 ft. 6 in., and they are continued over the main towers on each side of the central span, and pass below the ground, through inclined shafts excavated in the rock, to the anchorages.

The sectional area of steel wire in each rope is 2 sq. in., and the total sectional area in the six cables is 84 sq. in. The arrangements for anchoring the cables to the rock are shown in Plate V.*b*, and consist of six hollow cast-steel beams, 15 in. in diameter and 2½ in. thick, arranged three in each of two chambers excavated in the rock. An inclined rectangular shaft gives access to each of the chambers referred to from the ground level, and the cables pass down these to cast-iron thimbles, around which the ropes are coiled, and the ends of the wires are spread out in conical holes, where they are firmly secured by driving in round steel taper pins. The width between the cables over the piers is 48 ft. centre to centre, and in the centre of the bridge 32 ft. centre to centre. The towers are 112 ft. high, and the cables discharge their weight upon expansion rollers 3 ft. in diameter, which rest upon granite bed-stones, from which the pressure is distributed uniformly over the towers.

The stiffening girders, which are designed to prevent the main cables altering their curvature when the bridge is traversed by a moving load, are hinged at the centres, and spaced 15 ft. centre to centre across the bridge.

---

[1] "Suspension Bridges: A Study," *Trans. Am. Soc. C.E.*, 1896.

[2] *Trans. Am. Soc. C.E.*, 1896, p. 420.

[3] North Sydney Suspension Bridge. Engineers, J. E. F. Coyle and the author.

The main girders are 12 ft. 6 in. effective depth, and the cross-girders are 2 ft. deep. The method of connecting the cross-girders to the main girders, and the cables, and the sections of these girders, are sufficiently illustrated in Plates V.*a*, V.*b*.

Calculations.—The total dead load, including cables, girders, deck, bracing, handrails, etc., on central span is 0·93 ton per lineal ft.

The live load for a crowd of people, at 40 lbs. per sq. ft. of deck, is 0·50 ton per lineal ft.

The live load for a tram motor and three loaded cars is equivalent to a load of (say) 0·5 ton per foot run over 150 ft.

The cables must be strong enough to carry the total live load of 0·93 + 0·5 = 1·43 ton per lineal ft.; say 1·5 ton.

$$\text{Stress in centre} = \frac{\overline{500}^2 \times 1{\cdot}5}{8 \times 38{\cdot}5} = 1218 \text{ tons.}$$

$$\text{Stress at towers} = \sqrt{375^2 + 1218^2} = 1275 \text{ tons.}$$

Hence the maximum intensity of working stress in the cables is—

$$\frac{1275}{84} = 15{\cdot}2 \text{ tons per sq. in.}$$

$$\text{or a factor of safety of } \frac{100}{15{\cdot}2} = 6{\cdot}58.$$

This factor of safety will be slightly reduced by a fall in temperature, which diminishes the central deflection, and increased by a rise in temperature.

It can easily be shown, by equating the moment of resistance of the hollow steel girders at the anchorages with the bending moment produced by the maximum pull on the cables, that the sizes mentioned are sufficient for their purpose.

Again, the stability of the towers against a wind pressure of 30 lbs. per ft. can easily be shown to be ample.

The stiffening girders receive their maximum stresses during the passage of a tram motor with three loaded cars, and, as the tram line is shown on one side, so as to leave room for a carriage way on the other, the stiffening girder nearest the centre of the tram line will receive twice as great a load as the other.

The maximum bending moment in the stiffening girder for a live load of 0·5 ton per foot run is—

$$M = \frac{wl^2}{53} = \frac{0{\cdot}5 \times 500 \times 500}{53} = 2360 \text{ foot tons.}$$

The maximum negative bending moment is—

$$M = -\frac{wl^2}{64} = -\frac{0{\cdot}5 \times 500 \times 500}{64} = 1953 \text{ foot tons.}$$

The area required in the top and bottom chords to resist the alternating stresses due to the positive and negative bending moments may be found by taking the safe stress at 3 tons per sq. in.

Then—

$$\frac{2360}{12 \times 3} = 66 \text{ sq. in. about.}$$

The area in girder A at the point where the maximum moment occurs should be 22 sq. in., and that of B, 44 sq. in. The maximum shear is—

$$\frac{wl}{6} = \frac{0{\cdot}5 \times 500}{6} = 42 \text{ tons about.}$$

The maximum stress in the lattice bars is—

$$42 \times 1{\cdot}414 = 59{\cdot}4 \text{ tons.}$$

Only one bar will be in action at a time as they are designed to resist tensile stresses only. So that the area required allowing a stress of 6 tons per sq. in. is—

$$\frac{59{\cdot}4}{6} = 10 \text{ sq. in.}$$

The actual sections provided in the bridge are sufficiently illustrated in Plates V., V.*a* and V.*b*.

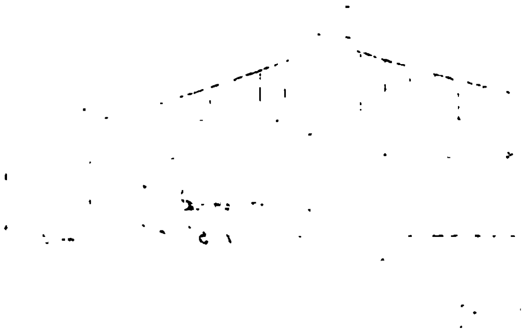
Stresses in the Cross-girders.—The longitudinal stiffening girders distribute the loads from the driving wheels of the tram motor over two or more cross-girders, and equalise the tensile stresses on the suspension rods, so that the live load upon the cross-girders must be considered in regard to the load upon the tram line estimated at half a ton per foot run, and the crowd of people estimated at 40 lbs. per sq. ft. of the deck not occupied by the tram. The dead load consists of the weight of the deck floor, handrailing, longitudinal timber beams, portion of stiffening girders and cross-girders.

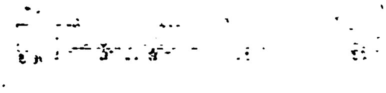
It may be shown that the maximum vertical pull on any suspension rod is about 11 tons, and the stress in the rod about 11½ tons. The suspension rods are 2 in. in diameter, and the effective area of the two bolts passing through the clips on the cables should be equal to that of the suspension rods.

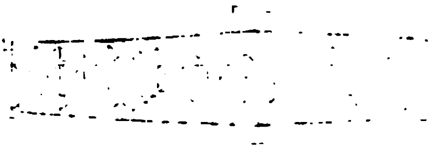
It may also be shown that the maximum intensity of working stress in the tension flange of the cross-girder shown in Plate V.*b* is 6 tons per sq. in.

Wind Bracing.—The maximum wind pressure over the whole bridge at 30 lbs. per sq. ft. acts horizontally over a surface equivalent to 500 ft. × 15 ft. = 7500 sq. ft. in the centre span, and 2250 and 1875 sq. ft. in the side spans. The total force in the central span is 7500 × 30 = 225,000 lbs., or (say) 100 tons. Since there are forty bays, the apex loads are 2½ tons.

The wind pressure is resisted partly by the cables and deck, and







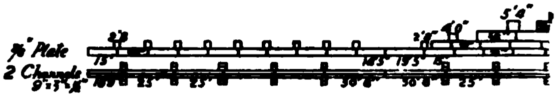
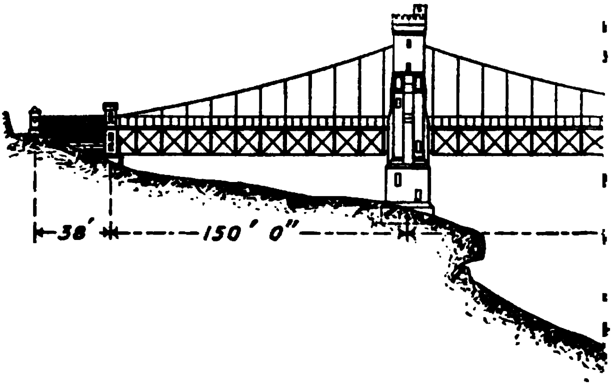


Plate
2 Channels
2'3"
15"
18'5"
19'5"
25"
30'6"
2'6"
4'0"
5'4"

DEVE


38'
150' 0"

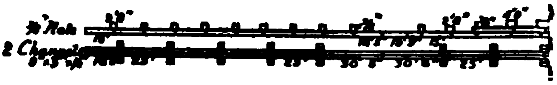
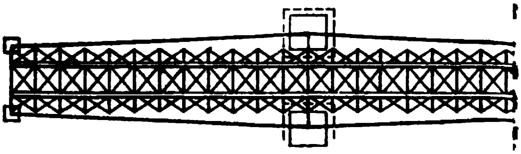
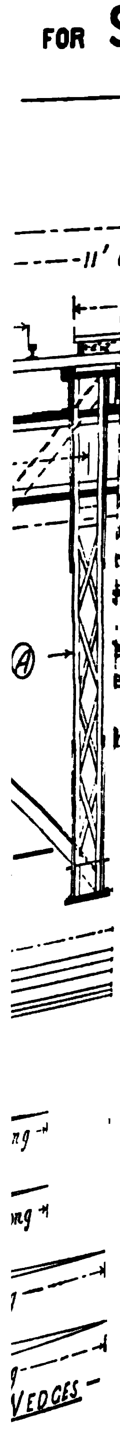


Plate
2 Channels
25"
30'6"

DEVE





artly by the wind bracing between the cross-girders and main girders. 1½ ton is supposed to act at points along the top chords of the tiffening girder corresponding with the positions of the cross-girders, will be found that the maximum stress in the end diagonals is 30 ons. The remaining stresses may be found in the same manner as an ordinary lattice girder.

The cross-girders act as struts, and the double system of diagonal racing will never act at the same time; one system will always be lack.

The remaining 1-ton apex load may well be left for the cables and e continuous deck of tallow-wood timber. This bridge is designed or light traffic, such as a horse tramway, which may be replaced by n electric tramway, and the live loads assumed in the foregoing calulations are greatly in excess of what is expected to pass over it. It s certainly not strong enough for ordinary town traffic. It should be regarded as a bridge of the minimum strength necessary to carry light raffic, such as that referred to, over a long span.

A three-hinged or two-hinged arched bridge would have been qually suitable for the site, and would have worked out quite as economically.

The suspension rods or hangers in the side spans might as well have been omitted, and the girders designed as detached spans.

APPENDIX I.

ARCH FIXED AT THE ENDS.

Mr. H. S. Mort, B.Sc., B.E., has extended the tables for vertical reactions and bending moment at the abutment, to provide for the other half of the arch (*i.e.*, for negative values of x). These are given in Tables A and B. He has also calculated a table for the bending moment at the quarter point when $a = \cdot 75$ (Table C), this being the value of a for a true parabolic arch.

Table A.—Vertical Reactions. x Negative.

x	·0	·01	·02	·03	·04	·05	·06	·07	·08	·09
– ·0	·5000	·4925	·4850	·4775	·4700	·4625	·4551	·4476	·4401	·4327
– ·1	·4252	·4178	·4104	·4030	·3957	·3883	·3810	·3737	·3665	·3592
– ·2	·3520	·3448	·3377	·3305	·3235	·3164	·3094	·3024	·2955	·2886
– ·3	·2817	·2749	·2682	·2615	·2548	·2482	·2417	·2352	·2287	·2223
– ·4	·2160	·2097	·2035	·1974	·1913	·1853	·1793	·1735	·1676	·1619
– ·5	·1562	·1507	·1452	·1397	·1344	·1291	·1239	·1188	·1138	·1088
– ·6	·1040	·0992	·0946	·0900	·0855	·0812	·0769	·0727	·0686	·0646
– ·7	·0607	·0570	·0533	·0498	·0463	·0430	·0397	·0366	·0336	·0308
– ·8	·0280	·0254	·0228	·0204	·0182	·0160	·0140	·0121	·0104	·0087
– ·9	·0072	·0059	·0047	·0036	·0026	·0018	·0012	·0007	·0003	·0001

TABLE B.—BENDING MOMENT AT THE ABUTMENT. x NEGATIVE.

x	·0	·01	·02	·03	·04	·05	·06	·07	·08	·09
– ·0	+ ·0625	·0650	·0674	·0697	·0719	·0740	·0761	0781	·0800	·0818
– ·1	·0835	·0851	·0867	·0882	·0896	·0909	·0921	·0932	·0942	·0951
– ·2	·0960	·0968	·0974	·0980	·0985	·0989	·0992	·0994	·0995	·0996
– ·3	·0995	·0994	·0992	·0989	·0985	·0980	·0957	·0969	·0962	·0954
– ·4	·0945	·0936	·0926	·0915	·0903	·0891	·0878	·0865	·0851	·0836
– ·5	·0820	·0804	·0788	·0771	·0754	·0736	·0718	·0699	·0680	·0660
– ·6	·0640	·0619	·0599	·0579	·0558	·0537	·0516	·0495	·0474	·0452
– ·7	·0430	·0409	·0388	·0367	·0346	·0325	·0304	·0284	·0264	·0244
– ·8	·0225	·0206	·0188	·0170	·0153	·0137	·0121	·0106	·0092	·0078
– ·9	·0065	·0054	·0043	·0033	·0025	·0018	·0011	·0007	·0003	·0001

x is the ratio $\frac{b}{c}$, corresponding to z in pp. 435 to 438.

BENDING MOMENT AT THE QUARTER POINT.

For a parabolic arch $a = {}$·75, in which case the expression reduces to—

$$\frac{c}{128}(1 + x^2)(5x - 1)(5 - x) - [c(x - \tfrac{1}{2})].$$

The expression on p. 438 may therefore be written—

$$\text{B.M.} = \frac{c}{128}(1 + x^2)(5x - 1)(5 - x) - [c(x - \tfrac{1}{2})] - \frac{15c}{32}(1 - x^2)^2(a - 0{\cdot}75),$$

the term $c(x - \frac{1}{2})$ occurring only when x lies between 0·5 and 1·0.

A table is given below for $a = {}$·75, and the term—

$$\frac{15c}{32}(1 - x^2)^2(a - 0{\cdot}75)$$

may be readily calculated from Table LXXVIII. for an arch not strictly parabolic.

TABLE C.—BENDING MOMENT AT ¼ POINT.

$$\frac{c}{128}(1 + x)^2(5x - 1)\ 5 - x) - [c(x - \tfrac{1}{2})].$$

	x	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09
Negative Values of Bending Moment.	− ·0	·0391	·0403	·0414	·0425	·0435	·0445	·0454	·0462	·0470	·0477
	− ·1	·0484	·0490	·0496	·0501	·0505	·0509	·0512	·0515	·0517	·0519
	− ·2	·0520	·0521	·0521	·0521	·0520	·0519	·0518	·0516	·0513	·0510
	− ·3	·0507	·0504	·0500	·0495	·0491	·0486	·0480	·0475	·0469	·0462
	− ·4	·0456	·0449	·0442	·0434	·0426	·0419	·0410	·0402	·0394	·0385
	− ·5	·0376	·0367	·0358	·0348	·0339	·0329	·0320	·0310	·0300	·0290
	− ·6	0280	·0270	·0260	·0250	·0240	·0230	·0220	·0210	·0200	·0190
	− ·7	·0180	·0171	·0161	·0152	·0142	·0133	·0124	·0116	·0107	·0099
	− ·8	·0090	·0083	·0075	·0068	·0061	·0054	·0048	·0041	·0036	·0030
	− ·9	·0025	·0021	·0017	·0013	·0010	·0007	·0004	·0002	·0001	·0000
	+ ·0	·0391	·0378	·0364	·0350	·0335	·0320	·0304	·0287	·0269	·0251
	+ ·1	·0232	·0212	·0191	·0170	·0148	·0125	·0102	·0077	·0052	·0026
Positive Values.	+ ·2	·0000	·0027	·0056	·0085	·0114	·0145	·0176	·0209	·0242	·0276
	+ ·3	·0310	·0346	·0382	·0419	·0458	·0497	·0536	·0577	·0619	·0661
	+ ·4	·0704	·0749	·0794	·0840	·0886	·0934	·0983	·1032	·1083	·1134
	+ ·5	·1187	·1140	·1094	·1049	·1005	·0962	·0919	·0878	·0838	·0793
	+ ·6	·0760	·0722	·0686	·0650	·0616	·0582	·0549	·0517	·0486	·0456
	+ ·7	·0427	·0399	·0372	·0346	·0321	·0296	·0273	·0258	·0229	·0209
	+ ·8	·0189	·0171	·0153	·0137	·0121	·0106	·0093	·0980	·0068	·0057
	+ ·9	·0047	·0038	·0030	·0023	·0017	·0012	·0008	·0004	·0002	·0000

Bending moment is negative except when x lies between + 0·2 and + 1·0.

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LOAN PERIOD 1	2	3
4	5	6

ALL BOOKS MAY BE RECALLED AFTER 7 DAYS

DUE AS STAMPED BELOW

DUE NRLF	MAR 26 1985	
MAY 15 1986		
AUTO DISC OCT 20 1987		

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